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Statistical Fracture of Brittle Materials

Progress on J.O. 5628 During FY 75 and Plans for FY 76

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The reasons why an improved understanding of the statistics of failure of brittle materials is important to the armed services are briefly reviewed. In addition, the status of work at Aerospace prior to undertaking the present work, which is supported by the Office of Naval Research and the Air Force Office of Scientific Research, and the technical proposal are reviewed. The tasks accomplished in FY 75 are discussed in some detail in Section II, subject to the limitation that this report was prepared two months prior to the end of the fiscal year. The major items of interest are: (1) a literature NEXT

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search for fracture data useful for validating or correcting theories, (2) recent discoveries that relate the theory for surface-distributed cracks to that for volume-distributed cracks, (3) work on the graphite model of Buch, Zimmer, and Meyer, (4) effects of porosity and crack interaction, and (5) the significance of asymptotic forms (such as the Weibull theory) in extreme value statistics. In Section III, the plans for FY 76 are discussed.

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I. INTRODUCTION

Because of their unique properties, brittle materials must be used in many military applications. Such materials have superior strength at high temperatures and are therefore used in reentry nosetips, e.g., graphite and beryllium. Investigation of the feasibility of using silicon nitride and silicon carbide for turbine motors is under way by DARPA. Also, virtually all materials that are transparent in either visible or infrared wavelengths are brittle. These, therefore, must be used in radomes and laser windows, e.g., quartz, silicon, zinc selenide, and cadmium telluride.

Brittle materials usually fail by fracture. Unlike ductile materials, they exhibit a wide dispersion in fracture stress, and this dispersion must be taken into account in design. This requires the use of fracture statistics, which are currently inadequately understood.

Fracture statistics can be obtained in the laboratory for simple tension and for some biaxial loading conditions. However, the stresses encountered in service are generally triaxial. These cannot be reproduced in the laboratory, and even if they could, the need for statistical data at each of a large number of combinations of stress would make testing prohibitively expensive. Thus, there is a need for a theory that shows how to use laboratory data in simple tension to predict the fracture statistics associated with any combination of stresses.

Until very recently, the best tool to use was Weibull's weakest link theory (Ref. 1). This is a statistical theory that assumes that fracture is caused by the presence of flaws of unspecified nature. In this theory, the fracture data in simple tension are represented approximately by proper choice of three parameters in a formula given by Weibull. For some cases, a reasonable fit can be obtained with a two-parameter formula obtained by making one of the parameters, σ_u (the minimum stress that can cause fracture), equal to zero. Weibull gave a rule for deducing polyaxial stress

statistics from uniaxial and showed how to apply it in the simpler two-parameter formula. However, he did not treat the usual case where $\sigma_u \neq 0$, as in the case of graphite. Moreover, the rule for converting from uniaxial to polyaxial statistics must be regarded as intuitively plausible rather than theoretically or experimentally demonstrated. A very approximate technique for extending Weibull's approach to the case $\sigma_u \neq 0$ was proposed by Dukes (Ref. 2), but no experimental check on its accuracy was attempted. A major shortcoming in Dukes' approach is that his treatment is internally inconsistent.

In 1970, a reexamination of the fundamentals of fracture statistics was undertaken by Batdorf. His approach was a weakest-link theory in which extreme value statistics were applied, but it differed from Weibull's in two major respects. First, the flaws were assumed to be cracks, with the directional sensitivity of cracks to the applied stresses. Second, rather than a three-parameter approximate fit to the data, a Taylor series expansion was used to fit the data to arbitrary precision. The theory was applied to uniaxial and biaxial test data on POCO graphite and found to be in very good agreement (Ref. 3).

Unlike POCO, the advanced graphites of greatest interest for nosetip application are not isotropic, but exhibit a moderate degree of anisotropy. Therefore, the theory was extended to cover this case (Ref. 4). The theory has also been extended to cover the case of materials with surface-distributed cracks (Ref. 5). Application of the theory to extensive data on pyrex cylinders indicates that it is in better agreement with experiment than any other available theory.

The preceding theory permits the use of fracture data in simple tension in order to determine the probability of failure of any volume of material in an arbitrary stress state. It can therefore be incorporated into a finite-element computer code for determination of the probability of survival of a nosetip during reentry.

An ideal statistical theory of fracture of brittle materials must take three things properly into account: (1) extreme value statistics, (2) fracture

mechanics, and (3) material properties and microstructure. Weibull's theory (Ref. 1) satisfies the first requirement approximately, but neither of the others. The later work described previously (Refs. 3 through 5) incorporates the first two requirements, but not the third. Very recently, efforts have been undertaken to develop statistical theories directed toward satisfying all three requirements.

Theories that do not satisfy the third requirement have to depend on experimental data in one stress state, e.g., simple tension, in order to establish the distribution of crack strengths; fracture statistics can then be determined for other stress states. The objective of theories that satisfy all three requirements is to predict the distribution of crack sizes from material microstructures. Such theories have many advantages: they require much less experimental data, they can be used to determine the changes in material processing needed for desired improvements in fracture behavior, and they should permit reliable extrapolation of the curve of failure probability versus stress to very low probabilities of failure.

Progress in the development of theories that satisfy all three requirements is being made in two areas. One is the statistical theory of fracture for brittle materials with preexisting integranular cracks. McClintock (Ref. 6) developed a two-dimensional model for such a material in which line cracks were all parallel and normal to the direction of tension. The corresponding three-dimensional theory for uniformly distributed penny-shaped cracks was developed by Batdorf (Ref. 7). This theory should apply to any polycrystalline ceramic. The second area is the theory of the initiation and growth of cracks in materials such as graphite. Such a theory, which predicts the stress-strain relation as well as the fracture stress, has been proposed by Buch, Zimmer, and Meyer (Ref. 8). In both cases, the theories should be extended and refined in order to improve their accuracy and to make them applicable to polyaxial stress conditions.

The prime objective of the present work is to further develop, refine, compare with experiment, and modify the theories as necessary - those in which the critical stress frequency distribution of cracks is determined experimentally,

and those in which the distribution is based on theoretical considerations. For convenience, the two classes of theory are referred to herein as experimentally based and theoretically based, respectively. The experimentally based theories will be of greater use in structural design in the short run. The theoretically based approaches have more ultimate potential and are of greater value to materials producers. They are also better when large changes in volume or failure probability are required in structural design.

In the proposal, the tasks that were specifically identified were the following:

1. Experimentally Based Approach

    a. Investigate the effects of porosity on fracture statistics. If possible, include porosity as a parameter in the theory.

    b. Reformulate the theory to include the statistical consequences of material variability.

    c. Extend the theory to include also predominantly compressive states of stress.

    d. Conduct an extensive literature search for fracture data with special emphasis on graphite. Check the validity of experiment by comparison with these data. Modify the theory if necessary.

2. Theoretically Based Approach

    a. Refine the theories for greater accuracy and to permit use of more realistic material models, e.g., variable grain size, porosity, and directional orientation.

    b. Extend the theories to apply to polyaxial stress conditions.

    c. Compare the theories with experiment and revise as necessary.

II. FY 75 ACCOMPLISHMENTS

A. LITERATURE SEARCH

There are two reasons why a literature search for fracture data is significant to this investigation:

1. By artful selection of data, or even innocent accidental choice, almost any theory can be corroborated or discredited. Only after an extensive survey can the experimental facts be stated with confidence.
2. The theory is fairly complicated, and refinements will probably be more complicated. It is not desirable to put too much effort into refinements until it is determined if the basic treatment is in accord with data.

The literature search for data and analyses of fracture statistics of brittle materials is progressing satisfactory. Western Research Contract Center has completed its computerized search for appropriate references. This search covered the NASA Data Bank Abstracts from 1969 to the present, Chemical Abstracts from 1970, and Engineering Abstracts back to 1971. Earlier sources are not in the computer banks and these are being located in the time-honored manual fashion by M. Adams at the Univ. of California, Los Angeles. He is digesting all relevant papers and reproducing appropriate graphs and tables for ready reference. This work is under way and will be completed before the end of FY 75. A sample page of the computer search report and a sample digest of a paper are presented in Appendix A.

B. FRACTURE STATISTICS OF BRITTLE MATERIALS WITH SURFACE CRACKS (Ref. 5)

The most important differences between this theory and the earlier one with volume-distributed cracks (Ref. 3) lie in the assumption that the surface crack planes are normal to the surface. Thus, the probability of failures in simple tension takes quite different forms for the two cases

$$P_f(\sigma) = 1 - \exp\left[-V\int_0^{\sigma} \frac{dN(\sigma_{cr})}{d\sigma_{cr}} \left(1 - \sqrt{\sigma_{cr}/\sigma}\right) d\sigma_{cr}\right] \tag{1}$$

$$P_f(\sigma) = 1 - \exp\left[-A\int_0^{\sigma} \frac{dN(\sigma_{cr})}{d\sigma_{cr}} \cos^{-1}\sqrt{\sigma_{cr}/\sigma}\; d\sigma_{cr}\right] \tag{2}$$

where Eqs. (1) and (2) apply to volume and surface-distributed cracks, respectively. In both theories, the function $N(\sigma_{cr})$, which represents the number of cracks per unit volume (area) with a critical stress equal to or less than σ_{cr}, is found by matching uniaxial test data. Knowledge of $dN/d\sigma_{cr}$ permits evaluation of corresponding expressions for probability of fracture under an arbitrary polyaxial state Σ by using

$$P_f(\sigma) = 1 - \exp\left[-V\int_0^{\sigma} \frac{\Omega}{4\pi} \frac{dN}{d\sigma_{cr}}\, d\sigma_{cr}\right] \tag{3}$$

$$P_f(\sigma) = 1 - \exp\left[-A\int_0^{\sigma} \frac{\omega}{\pi} \frac{dN}{d\sigma_{cr}}\, d\sigma_{cr}\right] \tag{4}$$

where Ω is a solid angle that corresponds to orientations in space, and ω is a planar angle that corresponds to orientations in a plane.

In Ref. 3, the technique used to solve Eq. (1) for $dN/d\sigma_{cr}$ was to expand it as a Taylor series and evaluate the coefficients so as to fit the fracture data in simple tension at a selected number of stress levels. This required truncating the series and inverting a matrix. It has subsequently been discovered that $dN/d\sigma_{cr}$ can be obtained from either Eq. (1) or Eq. (2) by solving them as integral equations. This avoids truncation errors as well as some danger of numerical instabilities and is therefore preferable. The solution of Eq. (1) is given in Appendix B.

The corresponding solution of Eq. (2) has also been developed and incorporated into a revised version of Ref. 5, which will be published later. However, a very surprising fact was recently discovered: for given uniaxial failure statistics $P_f(\sigma_x, 0)$, the predicted biaxial fracture statistics $P_f(\sigma_x, \sigma_y)$ are the same when either Eqs. (1) and (3) or Eqs. (2) and (4) are used; i.e., the volume theory and surface theory lead to the same polyaxial stress failure result, although, of course, the functions $VN(\sigma_{cr})$ and $AN(\sigma_{cr})$ are very different. Thus, for the sake of economy of analytical techniques and computer programs, the report is being revised to exploit the already-developed volume theory approach. This will not change the numerical results, but will constitute quite a change in the point of view and the depth of understanding of the problem.

C. GRAPHITE MODEL

The theories discussed in Refs. 1 through 7 assume a brittle material with preexisting randomly oriented cracks that remain unchanged during the loading process until the critical stress of the weakest crack is exceeded. At that time, the crack becomes unstable and grows without limit; as a result, the specimen fails.

An ingenious material model for graphite behavior has been proposed (Refs. 8 and 9) in which the cracks are visualized as being created by the loading process. This theory considers each grain in a typical commercial graphite as a crystal and exploits the fact that a crystal of graphite is very weak in the c-direction. Thus, random aggregations of grains that all have approximately the same c-orientation will easily fracture when a tensile stress is applied in that direction. An increase in stress will result in a relaxation in the degree of orientation needed for crack-like behavior and, therefore, more and larger cracks. The nonlinearity in the stress-strain curve is attributed to the increasing compliance as the cracks increase in number and size. Fracture of the material occurs when the radius r of the largest crack exceeds the critical size for the applied stress as given by the fracture mechanics equation

$$\sigma_{cr} = \frac{K_{Ic}}{2}\sqrt{\frac{\pi}{r}} \qquad (5)$$

The general concepts as outlined in the preceding paragraph are probably basically correct. Even though conceptually simple, a precise treatment of this model is mathematically intractable, and approximations are required. The theory has been successful in accounting approximately for the uniaxial stress-strain and fracture characteristics of typical graphites. In an effort to improve the accuracy, some work has been done in the present investigation. The main innovations are (1) the derivation of analytical expressions for results, where practicable, instead of depending on numerical analyses by computers, and (2) a revised criterion for crack size. In the theory of Refs. 8 and 9, an oriented penny-shaped array of N grains behaves like a crack only when the component of the applied stress in the c direction of <u>every</u> grain in the array exceeds the critical stress of the grain. In effect, this gives an array a strength equal to that of the most misaligned grain. In the present investigation, it is assumed that the strength of the array is equal to the average strength of the constituent grains. An analysis based on this revised assumption is given in Appendix C. This analysis includes a treatment of porosity for the case where pores are small, i.e., approximately grain-sized.

The revision has resulted in improved agreement with experiment, but there are some uncertainties in the legitimacy of some of the approximations used that need to be cleared up before publication. One concerns the elastic modulus of a material with uniformly distributed, partially oriented cracks such as those implied by the theory. Another is the legitimacy of the neglect of crack interaction, without which the analysis becomes exceedingly complex. The latter question is discussed in Section IVE.

D. APPROXIMATE METHODS OF DETERMINING THE FRACTURE STATISTICS FOR POLYAXIAL LOADING

In his original treatment of the statistics of fracture, Weibull (Ref. 1) introduced a two-parameter functional form for the relationship between

simple tension and probability of fracture. He also showed how to compute the corresponding statistics of failure for bending, torsion, and other stress states that involve only uniaxial tensile stresses. These applications have become well known and are widely used.

In problems that involve biaxial or triaxial tensile stresses, the situation is more complicated. Weibull gave, without formal proof, a procedure for treating such problems and showed how to apply it in some simple cases. Some investigators have expressed doubts concerning the rigor of Weibull's treatment of polyaxial stress status, and there are indications that these doubts were later shared by Weibull himself. Since, in addition, rather tedious calculations are required for each polyaxial stress state, in practical structures that involve continuously varying stress states, there is a natural tendency to use approximations. One simple approximation, which constitutes, in fact, the only technique suggested for handling polyaxial stress statistics in a recent treatise on fracture (Ref. 10) is to assume that

$$P_s(\sigma_1, \sigma_2, \sigma_3) = P_s(\sigma_1)P_s(\sigma_2)P_s(\sigma_3) \tag{6}$$

where P_s is the probability of survival, and c_1, σ_2, σ_3 are the principal stresses.

The analysis in Appendix D points out that the above approximation is often seriously unconservative. An exact[1] treatment is also given of failure statistics in biaxial tension for cases where the uniaxial tensile properties obey Weibull's two-parameter formula. In addition, a conservative approximation is given that is quite accurate in many cases of practical interest.

One application of the conservative approximation is the failure statistics of laterally loaded Zn Se disks. The data in question were obtained by Univ. of Dayton Research Institute (UDRI) as part of the Air-Borne Laser Laboratory program at AFWL. In Fig. 1, the unconservative nature of the prediction

[1]Exact from the standpoint of theories in Refs. 1 and 3.

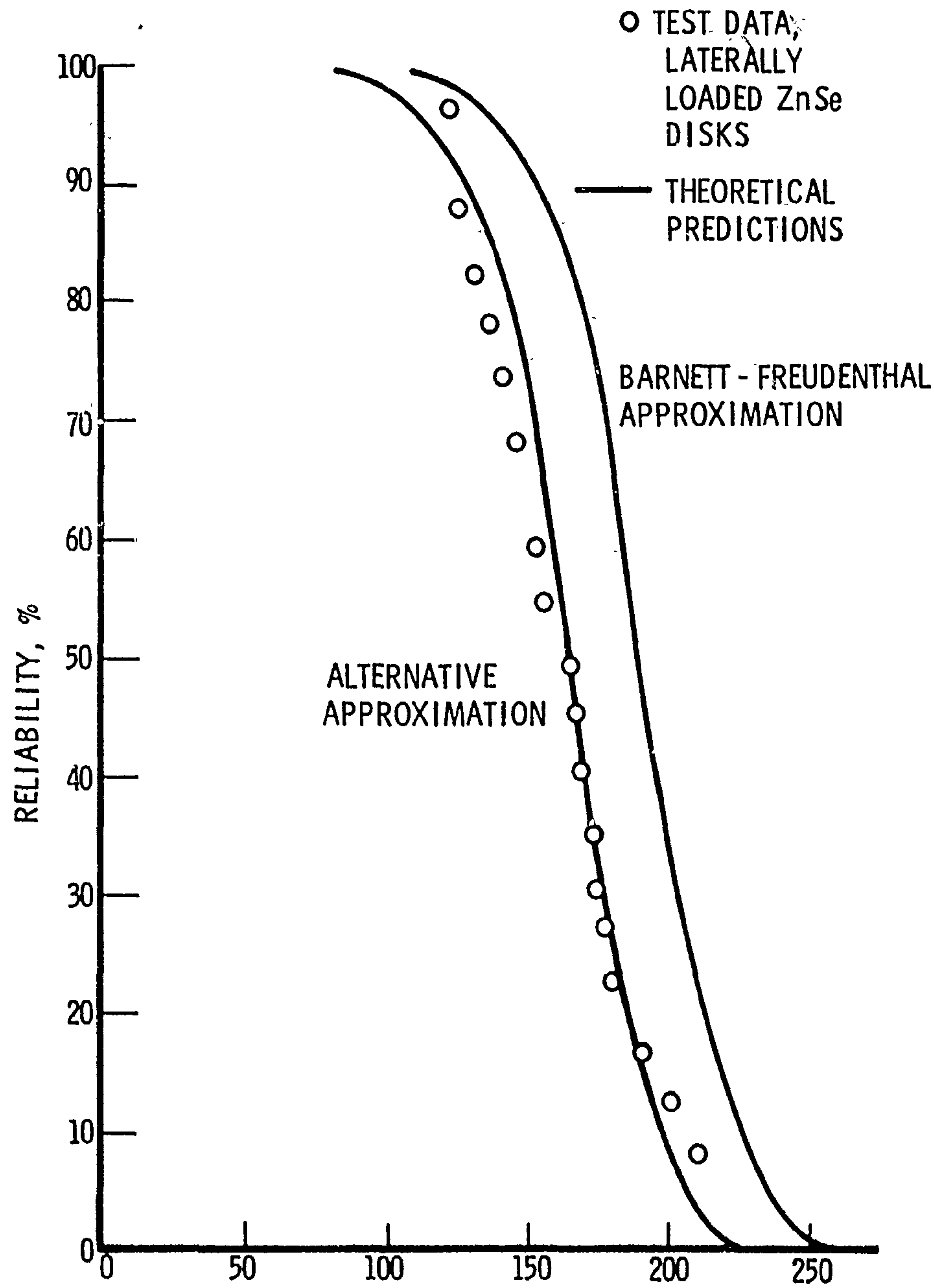

Figure 1. Comparison of Two Approximate Methods for Calculating the Reliability of Laterally Loaded Disks

based on Eq. (6) is confirmed. The approximation proposed in Appendix D appears to be much better.

E. STATISTICAL MODELS FOR BRITTLE FRACTURE (R. H. Huddlestone and D. C. Pridmore-Brown)

Two aspects of the statistics of brittle fracture have been studied: (1) the relationship between extreme-value statistics and possible fracture models, and (2) the combinatorial problem of calculating the flaw-size distribution in a material specimen that is assumed to consist of a regular array of identical grains, each with an equal and independent probability of being cracked. The status of this work is summarized below. Derivation of the crack-size distribution is discussed in more detail in Appendix E.

According to Gumbel (Ref. 11), the first application of the asymptotic theory of extreme values to fracture problems was made by Epstein (Ref. 12). In this and later work, the assumption was made that the breaking strength of the material sample is a linear function of the size of the largest flaw alone. This assumption, which is claimed to have gone unnoticed by other workers, according to McClintock, was justifiably criticized by him in a paper (Ref. 6) in which he developed an idealized model that does not correspond to an asymptotic distribution of extreme-value theory. McClintock's position was further corroborated by Batdorf (Ref. 7) with a more realistic three-dimensional model. Here, we try to clarify this situation and analyze the relevance of extreme-value statistics in this context.

If it is assumed that a material specimen under tension fractures at its weakest cross section, then a fracture test corresponds directly to selecting that cross section of smallest breaking strength from the large sample of cross sections in the specimen. In other words, the breaking strength of the specimen can be expected to be an extreme-value statistic irrespective of the inadequacy of the above linearity assumption.

If the breaking strength exhibits an asymptotically stable distribution as the length of the test specimens is increased, then, necessarily, by Gn'edenko's theorem of order statistics, the distribution will be the third

asymptotic one. It should be realized that this distribution was introduced on a purely phenomenological basis by Weibull (Ref. 1) and, consequently, is frequently referred to as the Weibull distribution. However, it should be emphasized that the present conclusions and their justification are not based on Weibull's model, but rather on a theorem of mathematical statistics.

There appears to be no compelling reason why the distribution of breaking strengths should be asymptotically stable on physical grounds. It is this point that dictates the applicability of the statistical conclusions stated above. Gumbel and Gn'edenko in fact give examples of extreme-order statistics that are not asymptotically stable and whose distribution does not approach any of the three asymptotic laws. The models of McClintock and of Batdorf are both of this type, and the rescaled shape of the distribution changes with the size of the specimen.

F. FLAW-SIZE MODEL BASED ON THEORY OF RUNS (R. H. Huddlestone and D. C. Pridmore-Brown)

The work on the flaw-size distribution is an effort to develop a mathematical model that will permit an explicit treatment and, therefore, a quantitative assessment of the errors implicit in the more heuristic models that are currently used. Preliminary theoretical analysis, supported by small specimen calculations made with the Aerospace on-line APL system, convinced us that many of the approximate estimates of flaw-size distribution in current use may involve large errors in certain applications.

We eventually noticed that the theory of runs, a well-developed branch of probability theory, would permit a direct calculation for the highly over-idealized one-dimensional case. Such a prototype could then be compared with the heuristic formulas at least in one dimension and could prossibly be used to improve two- and three-dimensional intuition as well. In any weakest link theory that relates strength to crack size, statistical questions concerning the distribution of crack size become important. Such questions are difficult to answer in general, and it has been common practice to make various simplifying assumptions. One is the assumption that crack interaction is negligible. Intuitively, this assumption appears valid in the limit where the

probability of finding a crack of a given size is very small, but it is not clear just when it breaks down.

In order to study this question, a simple model is considered in order that it can be treated analytically without neglecting crack interactions. In this model, the material is represented by a single row of grains, each with a given probability p of being cracked. A crack is defined as a row of one or more contiguous cracked grains, and we consider the probability that, in a sample of size n, i.e., in a row of n grains, there exists one or more cracks of size J or greater.

The mathematical theory that permits conclusions merely from the order in which the elements of the sample appear is called the theory of runs and is treated in books on statistics. In particular, some progress toward answering the above question was made by Mood (Ref. 13). However, he calculated the joint probability distribution for all crack sizes less than J and the total number of cracks $\geq$J. Thus, for even moderate values of J, his result involves a very large number of parameters that are irrelevant for our purposes and make it unsuitable for calculation. In Appendix E, an analysis is given in which we sum over all but one of the unwanted parameters and obtain

$$P(J) = \sum_{i=1}^{[pn/J]} (-1)^{i+1} \binom{qn+1}{i} \binom{n - Ji}{qn} \Big/ \binom{n}{qn} \qquad (7)$$

for the probability of finding a crack of size $\geq$J in a sample of n grains when the probability of cracking for each grain is p = 1 - q.

This result may be compared with a theory that neglects crack interaction, which, in this case, would yield

$$P_o(J) = 1 - (1 - p^J)^n \qquad (8)$$

A numerical comparison of these two formulas for a sample of size $n = 100$ and various crack sizes J is shown in Fig. 2. It is evident that the agreement is very poor for large or even moderate values of p.

In Appendix E, an asymptotic formula is derived for the exact expression (7) in the limit $n \to \infty$, provided $Jp^{J-1} \ll 1$

$$P(J) \sim 1 - (1 - p^J)^{n(1-p)} \tag{9}$$

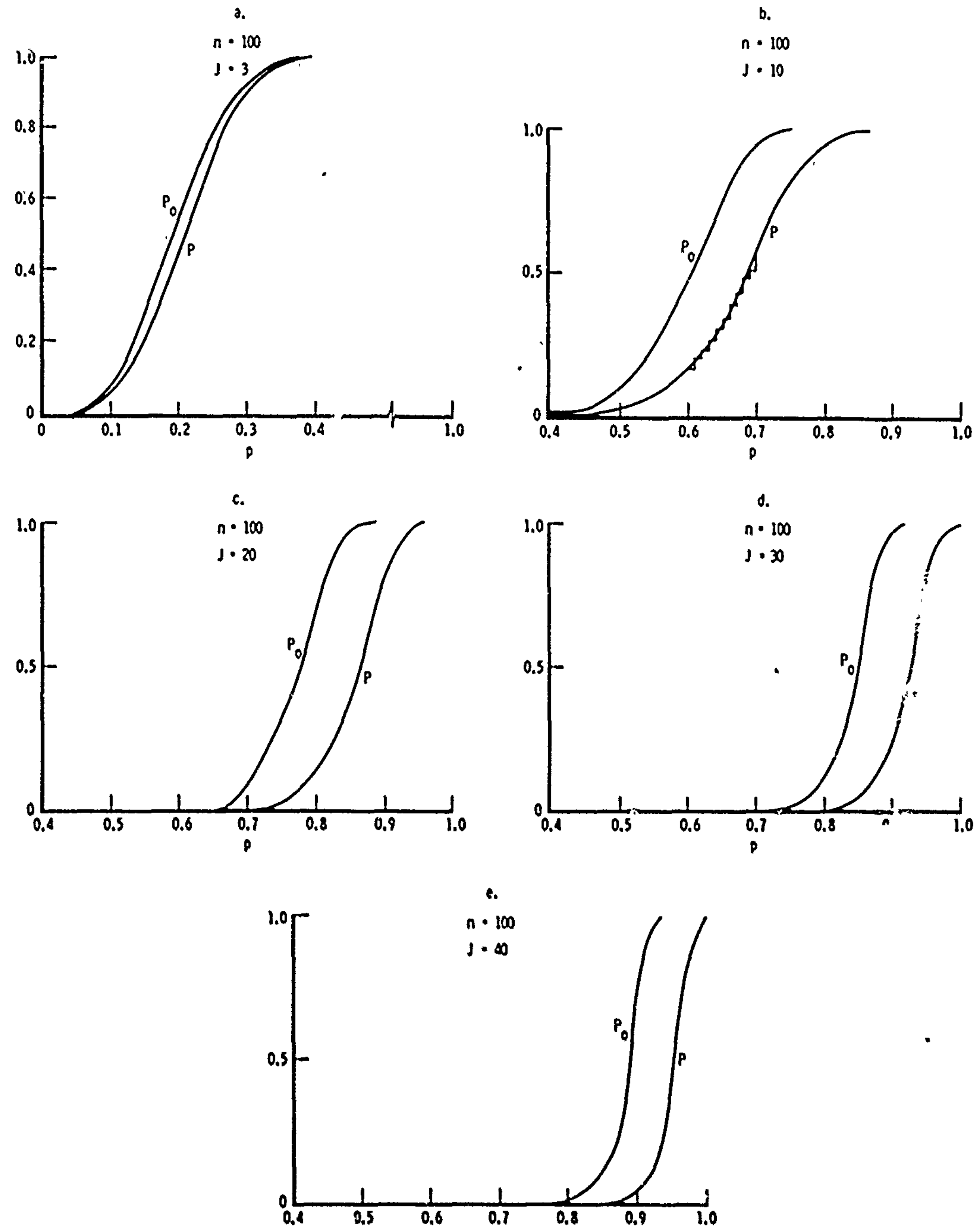

Figure 2. Numerical Comparison of Eqs. (7) and (8) for n = 100 and for Different Values of J

III. PLANS FOR FY 76

It is planned that most of the investigations discussed in Section II will be completed before 30 June 1975; those that are not will be completed in FY 76. It is planned that the identification of relevant published technical papers will be completed, appropriate data extracted, and the data processing started during FY 75. However, it will probably be many months before all the useful information is extracted from the large volume of data contained in the world's technical literature.

A. INCLUSION OF POROSITY AS A PARAMETER IN THE THEORY

This effort has been started for small pores in graphite (Appendix C). The approach followed is to assume that a pore is a missing grain and that then the load is shared equally by the remaining grains in the crack-like oriented array. For large pores, stress concentration factors must be considered. In effect, the neighboring grains will be subjected to a greatly increased stress and this will affect the statistics of failure.

B. REFORMULATION OF THE THEORY TO INCLUDE THE STATISTICAL CONSEQUENCES OF MATERIAL VARIABILITY

This task is of great practical importance. In normal engineering practice, if the material properties must be highly uniform, the rejection rate will be high and costs will increase. Even in the laboratory, it is nearly impossible to obtain identical specimens, and material variability obscures the effects under investigation.

C. EXTENSION OF THE THEORY TO INCLUDE PREDOMINANTLY COMPRESSIVE STRESSES

This task is much more difficult than the tensile stress task. In brittle materials subjected to tension, crack growth promotes instability and failure.

In compression, the crack grows in such a direction as to alleviate the stresses, and crack growth stops. Thus, material failure in compression is not a weakest-link phenomenon, but the end result of a lot of accumulated damage. Because of this, it is uncertain whether or not an adequate theory can be developed by the end of FY 76. However, the prospects are good because a large volume of new experimental information on compressive failure of alumina is being generated by M. Adams under the direction of G. Sines at the Univ. of California, Los Angeles.

D. EXTENSION OF THE THEORIES FOR APPLICATION TO POLYAXIAL STRESS CONDITIONS

For materials with preexisting cracks, in the approximation in which it is assumed that only the component of stress normal to the plane of the crack contributes to fracture, the polyaxial case has already been treated. In fact, however, the shear stress also contributes; thus, the above approximation is unconservative. For a Griffith crack (elliptical cylinder stressed in the plane normal to the cylinder axis), the contribution of the shear stress to fracture can be evaluated exactly and the fracture statistics revised accordingly. The objective for FY 76 is to go a step beyond this, if possible, and develop a statistical treatment for penny-shaped cracks.

E. REFINEMENT OF THEORIES FOR GREATER ACCURACY

This is, in a sense, an open-ended task. Refinements to include porosity and effects of shear have been identified earlier. It is expected that additional work, extending well beyond the work described in Appendix C, will be done on the graphite model of Buch, Zimmer, and Meyer. Variable grain size will be considered. To the extent possible, consideration will be given to the effects of crack interaction and variations in crack shape.

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7. S. B. Batdorf, Fracture Statistics of Brittle Materials with Intergranular Cracks, TR-0075(5450-76)-1, The Aerospace Corporation, El Segundo, California (10 October 1974).

8. J. D. Buch, "The Mechanical Properties of Graphites and Their Relation to a Physically Based Statistical Theory of Fracture," Proceedings of Conference on Continuum Aspects of Graphite Designs, CONF-701105, U.S. Atomic Energy Commission (1970), pp. 501-509.

9. J. D. Buch, J. E. Zimmer, and R. A. Meyer, An Analytical Microstructural Model for the Fracture of Graphite, ATR-74(7425)-4, The Aerospace Corporation, El Segundo, California (28 June 1974).

10. A. M. Freudenthal, "Statistical Approach to Brittle Fracture," Fracture, Vol. 2, ed. H. Liebowitz, Academic Press, New York (1968).

11. E. J. Gumbel, Statistics of Extremes, Columbia Univ. Press (1958).

12. B. Epstein, J. App. Phys. 19, 140 (1948).

13. A. M. Mood, "The Distribution Theory of Runs" Ann. Math. Stat. 11, 367 (1940).

APPENDIX A. SAMPLE COMPUTER SEARCH REPORT AND DIGEST

A sample page of the computer search of library sources is shown below. This is followed by sample pages of the digest of a paper that contains statistical fracture data.

123943a Fracture in polycrystalline ceramics. Coble, R. L.; Parikh, N. M. (Dep. Metall. Mater. Sci., Massachusetts Inst. Technol., Cambridge, Mass.). *Fracture* 1972, 7, 243-314 (Eng). Edited by Liebowitz, Harold. Academic: New York, N. Y. A review is given on the fracture in polycryst. ceramics, the plastic and brittle behavior of ceramic oxides, fracture in polycryst. Al_2O_3, and factors affecting strength. 130 Refs.

124020c Evaluation of the brittleness of refractories. Pisarenko, G. S.; Gogotsi, G. A. (Inst. Probl. Prochn., Kiev, USSR). *Ogneupory* 1974, (2), 44-7 (Russ). A criterion, χ, for the brittleness of refractories is proposed: $\chi = \epsilon_y/\epsilon_p$ where ϵ_y is the elastic and ϵ_p the limiting strains, resp. A material is subject to brittle fracture when χ is equal or close to unity. χ Decreases with increasing temp.; the decrease is pronounced in refractories of the aluminosilicate type and negligible in SiC. T. Ordentlich

137844z Influence of specimen size and mode of loading on the fracture of graphite. Marshall, P.; Priddle, E. K. (Berkeley Nucl. Lab., Cent. Electr. Generating Board, Berkeley/Gloucestershire, Engl.). *Carbon* 1973, 11(6), 627-31 (Eng). Predictions of graphite strength under tensile and bending loads made by using fracture mechanics were compared with new exptl. data and with calcns. by the method of W. Weibull (1939). The methods are comparable for similar specimens, but the Weibull method failed to predict the obsd. effects for large specimen-size differences. The basis of the Weibull theory (a volumetric flaw distribution) might be inapplicable to failure of materials resulting from surface defects. Examn. of graphite rods showed a defect-size distribution across the material that led to prediction of lower strengths for the inner rod regions with respect to the outer regions. Exptl. results followed the predicted trend, but further examn. of the fracture-mechanics techniques must be undertaken before it can be recommended as a design method for graphite materials.

VOL.79,1973

96028v Strength and fracture properties of glass-ceramics. Hing, P.; McMillan, P. W. (Dep. Phys., Univ. Warwick, Coventry, Engl.). *J. Mater. Sci.* 1973, 8(7), 1041-8 (Eng). The effect of an isothermal heat treatment on the strength and fracture surface energy of a glass-ceramic derived from a Li silicate glass was studied. The strength and effective surface energy for crack initiation increase as the mean free path in the intercryst. glass decreases. The strengthening and toughening mechanisms are discussed.

96102q Effect of air pressure on the brittle failure of graphite. Dubrovskii, K. E.; Kissel, V. V. (Moscow, USSR). *Fiz.-Khim. Mekh. Mater.* 1973, 9(1), 33-5 (Russ). The ultimate tensile stress on static bending (σ_b) of graphite sealed off from the pressure air increases linearly with the pressure <300 atm. A steep increase of σ_b was obsd. at higher pressures. The course of the curve at <300 atm. follows the Mirolyubov law (I. L. Gol'denblat and V. A. Kopinov, 1968). If air were in contact with the test specimen, the function of σ_b vs. pressure was linear over the whole pressure range (<500 atm) and had a smaller slope than that in the previous case; the value of σ_b at 300 atm was lower by ~30%. The weakening effect of the air is due to the lowering of the surface energy of graphite resulting from the absorption of air by graphite. H. Kucova

96109x Effects of porosity on fracture of aluminum oxide. Coppola, J. A.; Bradt, R. C. (Mater. Sci. Dep., Pennsylvania State Univ., University Park, Pa.). *J. Amer. Ceram. Soc.* 1973, 56(7), 392-3 (Eng). The fracture surface energy was detd. for 1 μm grain size Al_2O_3 ceramics contg. 0.5% MgO with porosity <8%. The porosity was located primarily at grain boundaries while the fracture topography was predominantly intergranular. The fracture mode was mixed; ~20% more was transgranular. The absence of a distinct decrease in the fracture surface energy with decreasing porosity suggests a pore-crack front interaction.

127583a Hertzian fracture of a lithium silicate glass and glass-ceramic. Nadeau, J. S.; Rao, A. S. (Metall. Dep., Univ. British Columbia, Vancouver, B. C.). *J. Can. Ceram. Soc.* 1972, 41, 63-7 (Eng). The fracture energy of Li silicate glass prepd. as a homogeneous glass is less than that of the phase-sepd. glass which is less than that of the glass-ceramic.

AFML-TR-66-51

BIAXIAL-FRACTURE STRENGTH OF BRITTLE MATERIALS

H. W. BABEL

DOUGLAS AIRCRAFT COMPANY, INC.

TECHNICAL REPORT AFML-TR-66-51

MARCH 1966

AIR FORCE MATERIALS LABORATORY
RESEARCH AND TECHNOLOGY DIVISION
AIR FORCE SYSTEMS COMMAND
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13. ABSTRACT

The investigation described herein was conducted to develop an analytical model to accurately predict the biaxial-fracture strengths of brittle materials. Upon the basis of work originated by Inglis and Griffith, a new biaxial-failure criterion has been developed. The new criterion is a flaw-based theory concerned with the influence of spherical and ellipsoidal discontinuities upon the fracture strength of brittle materials. Also considered in the new theory are the sharp discontinuities that constitute a special limiting condition of the new theory which coincides with Griffith's criterion.

An experimental investigation was conducted to demonstrate the validity of the new biaxial-failure criterion. One-hundred and eighteen porous zirconia test specimens were used for the fracture tests. These tests were conducted for uniaxial tension and compression and for two biaxial-stress states in the tension-compression quadrant.

A careful analysis was made to develop test conditions that largely circumvented the experimental difficulties encountered by previous investigators. Photoelastic and strain-gage studies were conducted to verify the experimental techniques; the results showed that homogeneous stress fields were developed in the gage area of the test specimens.

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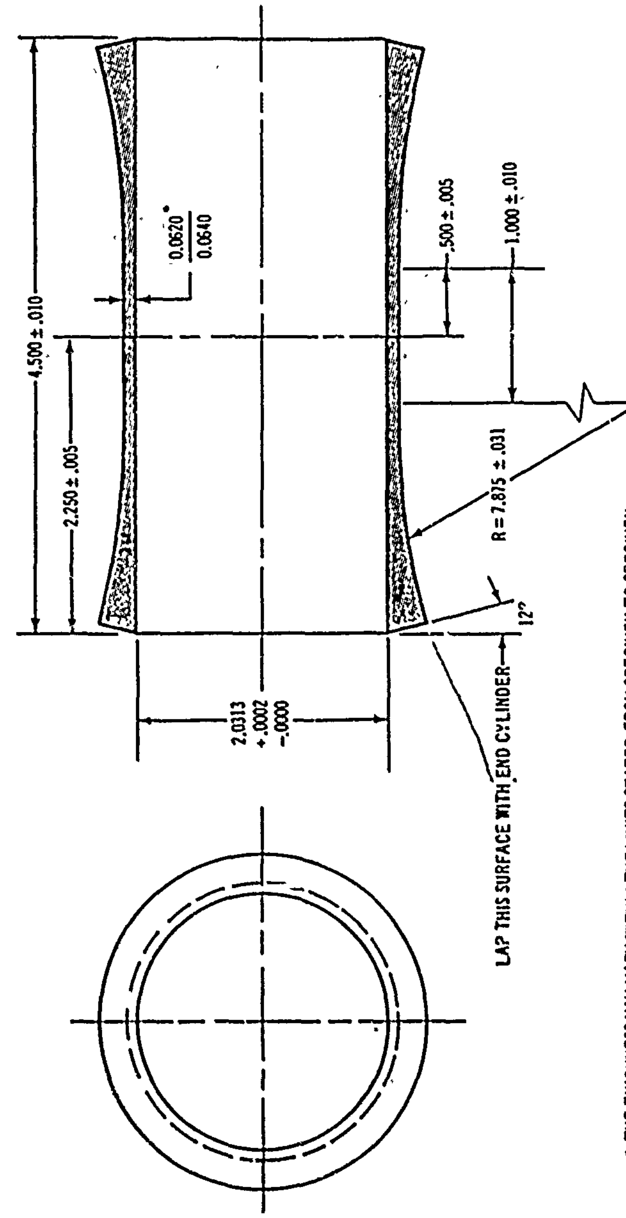
The experimental data were compared to the predictions of the most prominent failure criteria. The limited data available in the literature tended to fit the new criterion better than the maximum-normal-stress, the Coulomb-Mohr, the Griffith, or the Weibull biaxial-failure criteria. Throughout the entire compression-tension quadrant, the fracture stresses obtained from this program were in excellent agreement with the predictions of the new criterion; no such agreement was obtained between these data and the other failure criteria.

Testing procedures had a pronounced influence upon the test results. When the axial-compressive load component was applied before the hoop-tensile component, good agreement was obtained between the data and the new criterion, except in the lower region of the tension-compression quadrant. In this region, the experimentally determined strengths were higher than those predicted by the new criterion. These differences could be explained by a mechanism whereby favorably oriented cracks were closed by the compressive stresses. Such closure would permit stresses to be transmitted across crack boundaries, a condition inadmissible in Inglis' original solution.

The excellent agreement obtained between the predictions of the new criterion and the experimental data demonstrates the validity of the analytical model and the reliability of the experimental techniques. Such agreement supports the basic hypotheses used to develop the analytical model.



* THE THICKNESS MAY VARY WITHIN THE LIMITS STATED FROM SPECIMEN TO SPECIMEN, BUT ON ANY ONE SPECIMEN THE THICKNESS MUST NOT VARY MORE THAN ± 0.0002.

Figure 11. -- Test specimen

TABLE 18
TEST RESULTS FOR BATCH ONE

SPECIMEN NO.	TEST NO.	DIAMETER O.D. (in.)	DIAMETER I.D. (in.)	PRESSURE AT FAILURE (psig)	HOOP STRESS (psi)	AXIAL LOAD (lb.)	AXIAL STRESS (psi)	SUITABLE FOR ANALYSES[c]
				UNIAXIAL HOOP TENSION TESTS				
13	3	2 1548	2 0315	213 0	3620	....	....	Yes
14	1	2 1551	2 0315	....	....	....	....	No
23	5	2 1550	2 0314	247 0	4183	....	....	Yes
29	2	2 1551	2 0314	204 5	3460	....	....	Yes
Average					3754	....	....	....
Normalized Average[a]					1.0000	....	....	....
				-3:1 BIAXIAL TESTS[b]				
1	24	2.1551	2 0313	186 0	3145	4331	10,639	Yes
3	25	2.1552	2.0314	190 0	3213	4330	10 636	Yes
7	7	2.1550	2.0318	214 0	3636	4330	10 689	Yes
15	22	2.1548	2 0315	182 0	3101	4338	10,701	Yes
24	6	2.1550	2.0313	214 0	3624	4330	10 647	Yes
Average					3344	....	10,662	....
Normalized Average[a]					0 8908	....	-2.8402	....
				-5:1 BIAXIAL TESTS[b]				
8	16	2.1548	2 0320	....	....	....	....	No
10	17	2 1559	2 0314	151 0	2545	7240	17,624	Yes
12	15	2.1549	2.0320	....	....	....	....	No
26	23	2.1549	2 0320	96 0	1645	7250	17 835	Yes
28	21	2.1552	2.0318	119 0	2027	7250	17,866	Yes
Average					2072	....	17,775	....
Normalized Average[a]					0 5519	....	-4 7349	....
				UNIAXIAL COMPRESSION TESTS				
2	95	2.1545	2 0314	....	....	9 575	23 812	Yes
5	75	2.1550	2 0314	....		10,450	25,713	Yes
25	76	2.1552	2 0314	....	....	....	....	No
Average						....	24,763	....
Normalized Average[a]						....	-6 5964	....

[a] Stresses normalized with respect to the average uniaxial hoop stress.

[b] Test identification code, the codes -3 1 and -5.1 are not stress states.

[c] See table 5-2 for comments on each test.

TABLE 20

TEST RESULTS FOR BATCH TWO

SPECIMEN NO.	TEST NO.	DIAMETER O.D. (in.)	DIAMETER I.D. (in.)	PRESSURE AT FAILURE (psig)	HOOP STRESS (psi)	AXIAL LOAD (lb)	AXIAL STRESS (psi)	SUITABLE FOR ANALYSES[c]
				UNIAXIAL HOOP TENSION TESTS				
34	44	2 1545	2 0314	...	...	...	...	No
32	43	2 1551	2 0318	210 5	3565	...	...	Yes
44	33	2 1554	2 0316	194 5	3284	...	...	Yes
47	4	2 1552	2 0314	204 0	3420	...	...	Yes
Average					3423	...	...	...
Normalized Average[a]					1 0000	...	...	...
				-3:1 BIAXIAL TESTS[b]				
38	46	2 1540	2 0316	157 5	2693	4315	10 723	Yes
43	45	2 1555	2 0315	...	...	...	...	No
48	47	2 1549	2 0314	189 0	3203	4327	10 655	Yes
55	49	2 1550	2 0320	...	...	...	...	No
Average					2948	...	10 689	...
Normalized Average[a]					0 8612	...	– 3 1227	...
				-5:1 BIAXIAL TESTS[b]				
39	18	2.1553	2 0313	...	...	...	...	No
42	53	2 1551	2.0316	145 5	2466	7250	17 853	Yes
45	50	2 1551	2 0315	11 0	115	7250	17 840	Yes
50	19	2 1550	2 0320	152 0	2587	7250	17 923	Yes
53	20	2.1555	2 0315	...	...	...	...	No
58	55	2.1551	2 0320	109 0	1855	7200	17 782	Yes
Average					1755	...	17 849	...
Normalized Average[a]					0 5127	...	– 5 2144	...
				UNIAXIAL COMPRESSION TESTS				
41	79	2.1530	2 0315	...	...	10,000	25 044	Yes
51	87	2.1558	2 0316	...	...	9,900	24 235	Yes
57	74	2.1552	2.0314	...	...	8,650	21 272	Yes
Average							23 517	...
Normalized Average[a]							– 6 8703	...

[a] Stresses normalized with respect to the average uniaxial hoop stress.

[b] Test identification code, the codes -3 1 and -5 1 are not stress states.

[c] See table 5-4 for comments on each test.

TABLE 22
TEST RESULTS FOR BATCH THREE

SPECIMEN NO.	TEST NO.	DIAMETER O.D. (in.)	DIAMETER I.D. (in.)	PRESSURE AT FAILURE (psig)	HOOP STRESS (psi)	AXIAL LOAD (lb)	AXIAL STRESS (psi)	SUITABLE FOR ANALYSES[c]
UNIAXIAL HOOP TENSION TESTS								
61	8	2.1577	2 0315	196 0	3304			Yes
69	13	2.1554	2 0314	183 3	3094			Yes
73	11	2.1552	2 0315	180 3	3054			Yes
74	12	2.1557	2 0314	183.0	3085			Yes
77	9	2.1540	2 0315	196 0	3351			Yes
86	10	2.1549	2 0315	190 3	3228			Yes
88	14	2 1557	2 0314	201 0	3388			Yes
Average					3215			
Normalized Average[a]					1 0000			
-3:1 BIAXIAL TESTS[b]								
60	68	2.1558	2 0313	190 5	3203	4330	10576	Yes
62	29	2 1553	2 0216	180.0	3046	4330	10 644	Yes
63	26	2.1548	2 0314	162 0	2748	4330	10 673	Yes
64	28	2.1559	2 0314					No
82	30	2 1550	2 0315	156 0	2644	4330	10 662	Yes
87	31	2.1556	2 0314	166 0	2798	4330	10 602	Yes
89	52	2.1538	2 0314	139 0	2376	4330	10 760	Yes
Average					2803		10 652	
Normalized Average[a]					0 8719		– 3 3135	
-5.1 BIAXIAL TESTS[b]								
65	54	2.1551	2 0314	83 5	1413	7250	17 872	Yes
66	27	2 1549	2 0314	118 0	2002	7250	17 853	Yes
72	36	2 1552	2 0313	158 8	2670	7250	17,709	Yes
76	32	2.1557	2.0315	124 5	2055	7250	17 752	Yes
79	72	2.1558	2 0315	143.5	2421	7250	17 735	Yes
80	34	2.1548	2 0314	114 0	1934	7250	17,870	Yes
83	35	2.1549	2.0314	97.0	1646	7250	17,853	Yes
Average					2026		17,799	
Normalized Average[a]					0 6302		– 5 5362	
UNIAXIAL COMPRESSION TESTS								
67	48	2.1553	2 0316					No
68	86	2 1557	2 0313			9 475	23 166	Yes
70	78	2.1553	2 0314			9 275	22,766	Yes
78	93	2.1555	2 0316			10,125	24 847	Yes
81	88	2.1553	2 0314			9 375	22,244	Yes
84	94	2 1554	2.0316			8 775	21,555	Yes
85	77	2.1550	2 0315			9 000	22,162	Yes
Average							22,793	
Normalized Average[a]							– 7 0896	

[a] Stresses normalized with respect to the average uniaxial hoop stress.

[b] Test identification code; the codes –3,1 and –5,1 are not stress states.

[c] See Table 5-6 for comments on each test.

TABLE 24

TEST RESULTS FOR BATCH FOUR

SPECIMEN NO.	TEST NO.	DIAMETER O.D. (in)	DIAMETER I.D. (in)	PRESSURE AT FAILURE (psig)	HOOP STRESS (psi)	AXIAL LOAD (lb)	AXIAL STRESS (psi)	SUITABLE FOR ANALYSES[c]
				UNIAXIAL HOOP TENSION TESTS				
94	90	2 1525	2 0316	182 5	3161			Yes
97	89	2 1535	2 0315	186 0	[illegible]			Yes
108	97	2.1550	2.0314	167.0	2824			Yes
110	37	2.1557	2 0316	175 5	2961			Yes
113	42	2.1552	2.0356	169 0	2557			Yes
116	98	2 1552	2 0314	182 5	3086			Yes
118	99	2.1555	2 0314	182 5	3084			Yes
Average					3040			
Normalized Average[a]					1.0000			
				−3:1 BIAXIAL TESTS[b]				
91	66	2.1555	2 0314	148 5	2507	4330	10 610	Yes
96	60	2.1555	2 0315	149 0	2520	4330	10 618	Yes
100	56	2 1523	2 0314					No
102	57	2.1555	2 0326	156 0	2656	4330	10 713	Yes
104	59	2 1558	2 0314	163 0	2742	4330	10 584	Yes
105	58	2.1555	2 0314	143 0	2414	4330	10 610	Yes
107	70	2 1550	2 0315	160 5	2720	4330	10,662	Yes
Average					2593		10 633	
Normalized Average[a]					0 8530		−3 4977	
				−5:1 BIAXIAL TESTS[b]				
90	65	2 1550	2 0315	96 5	1637	7250	17,853	Yes
98	67	2 1558	2 0315	47 8	805	7250	17,735	Yes
103	61	2 1557	2 0315	135.5	2284	7250	17 752	Yes
106	69	2 1550	2 0314					No
109		2 1558	2.0315					No
111	64	2.1559	2.0324	104 0	1765	7250	17 844	Yes
119	63	2.1554	2.0314	47 0	793	7250	17,783	Yes
Average					1457		17,790	
Normalized Average[a]					0 4753		−5 8520	
				UNIAXIAL COMPRESSION TESTS				
93	80	2.1551	2 0314			8575	21 079	Yes
95	96	2.1558	2 0314			9975	24 383	Yes
99	81	2.1559	2 0315			9100	22 244	Yes
101	85	2 1557	2 0318			9375	23 006	Yes
112	83	2 1559	2 0315			9475	23 161	Yes
115	82	2.1555	2 0316			8925	21 902	Yes
117	84	2.1558	2.0313			7970	19,468	Yes
Average							22 178	
Normalized Average[a]							−7 2954	

[a] Stresses normalized with respect to the average uniaxial hoop stress

[b] Test identification code, the codes −3:1 and −5:1 are not stress states.

[c] See Table 5-8 for comments on each test.

TABLE 26

TEST RESULTS FOR BATCH FIVE

SPECIMEN NO.	TEST NO.	DIAMETER O.D. (in.)	DIAMETER I.D. (in.)	PRESSURE AT FAILURE (psig)	HOOP STRESS (psi)	AXIAL LOAD (lb)	AXIAL STRESS (psi)	SUITABLE FOR ANALYSES[c]
UNIAXIAL HOOP TENSION TESTS								
120	41	2.1552	2.0314	169 0	2857			Yes
123	38	2.1550	2.0314	186 0	3150			Yes
128	40	2.1554	2 0314	217.0	3663			Yes
129	39	2.1555	2 0314	209 0	3528			Yes
131	100	2.1551	2 0315	207 0	3510			Yes
133	101	2.1554	2.0315	200 0	3382			Yes
150	103	2.1552	2 0314	202.3	3420			Yes
Average					3359			
Normalized Average[a]					1 0000			
-3:1 BIAXIAL TESTS[b]								
132	51	2.1545	2 0316	176 0	2991	4330	10 715	Yes
134	71	2.1549	2.0314	161.5	2737	4330	10 662	Yes
Average					2867		10 689	
Normalized Average[a]					0 8535		-3 1921	
-5:1 BIAXIAL TESTS - STANDARD[b]								
122	62	2.1553	2 0314	137.0	2317	7250	17,796	Yes
130	73	2.1559	2.0313	139 0	2336	7200	17 574	Yes
142	110	2.1555	2.0315	115 5	1950	7250	17,778	Yes
146	111	2.1555	2 0320	147.5	2501	7200	17 725	Yes
Average					2276		17,718	
Normalized Average[a]					0 6776		-5 2748	
-5:1 BIAXIAL TESTS - PRESSURE APPLIED FIRST[b]								
125	105	2.1550	2 0315	120 0	2036	7450	18 345	Yes
126	106	2.1530	2 0315	120 0	2068	6550	16 404	Yes
127	109	2.1550	2 0315	121.0	2052	6725	16 560	Yes
135	104	2.1551	2.0314	120 0	2032	6750	16 593	Yes
Average					2047		16 976	
Normalized Average[a]					0 6094		-5 0539	
-5:1 BIAXIAL TESTS - CONSTANT STRESS STATE[b]								
136	113	2.1556	2 0316	133 5	2254	7400	18 146	Yes
144	112	2.1549	2 0316	128 0	2175	6975	17 205	Yes
147	114	2.1553	2 0315	128 0	2173	7025	17,256	Yes
149	115	2.1546	2 0315	111.3	1893	6100	15,073	Yes
Average					2124		16 920	
Normalized Average[a]					0 6323		-5 0372	
AXIAL COMPRESSION TESTS								
137	92	2.1548	2 0314			9400	23 170	Yes
138	107	2 1549	2 0314			9450	23 270	Yes
139	102	2.1553	2 0315			9600	23 581	Yes
143	108	2.1552	2 0314			9750	23 950	Yes
148	91	2 1554	2 0316			9650	23,704	Yes
Average							23 535	
Normalized Average[a]							-7 0065	

[a] Stresses normalized with respect to the average uniaxial hoop stress

[b] Test identification code, the codes -3.1 and -5.1 are not stress states

[c] See Table 5-10 for comments on each test.

APPENDIX B. INTEGRAL EQUATION SOLUTION FOR $N(\sigma_{cr})$

In the theory for volume-distributed cracks,[1] the equation for the probability of failure in simple tension is given as

$$P_f(\sigma) = 1 - \exp\left[-V\int_0^{\sigma}\left(1 - \sqrt{\sigma_{cr}/\sigma}\right)\frac{dN}{d\sigma_{cr}}\, d\sigma_{cr}\right] \qquad \text{(B-1)}$$

This can be regarded as an integral equation relating P_f and N, which can be solved as follows:

$$P_s = 1 - P_f = \exp\left[-V\int_0^{\sigma}\left(1 - \sqrt{\sigma_{cr}/\sigma}\right)\frac{dN}{d\sigma_{cr}}\, d\sigma_{cr}\right] \qquad \text{(B-2)}$$

$$\ln P_s(\sigma) = -V\int_0^{\sigma}\left(1 - \sqrt{\sigma_{cr}/\sigma}\right)\frac{dN}{d\sigma_{cr}}\, d\sigma_{cr} \qquad \text{(B-3)}$$

Integration by parts yields

$$\ln P_s = -V\left(1 - \sqrt{\sigma_{cr}/\sigma}\right)N(\sigma_{cr})\Big|_0^{\sigma} + \int_0^{\sigma} - \frac{V}{2}\ _0^{\sigma}\ \frac{N(\sigma_{cr})\, d\sigma_{cr}}{\sqrt{\sigma\,\sigma_{cr}}} \qquad \text{(B-4)}$$

[1] S. B. Batdorf and J. G. Crose, "A Statistical Theory for the Fracture of Brittle Structures Subjected to Nonuniform Polyaxial Stresses," J. Appl. Mech. 41 (2), 459 (1974).

Since $N(o) = 0$, the integrated term is zero at both limits and therefore vanishes. Multiplying both sides of the remaining equation by $\sqrt{\sigma}$ and then differentiating, we obtain

$$\frac{d}{d\sigma}\left(\sqrt{\sigma}\, \ln P_s(\sigma)\right) = -\frac{V}{2}\frac{N(\sigma)}{\sqrt{\sigma}} \qquad \text{(B-5)}$$

or

$$N(\sigma) = -\frac{2}{V}\sqrt{\sigma}\frac{d}{d\sigma}\left\{\sqrt{\sigma}\, \ln P_s(\sigma)\right\} \qquad \text{(B-6)}$$

APPENDIX C. SOME APPROXIMATE TREATMENTS OF FRACTURE STATISTICS FOR POLYAXIAL STRESS STATES

In his original treatment of the statistics of fracture, Weibull (Ref. C-1) introduced a two-parameter functional form for the relation between simple tension and probability of fracture. He also showed how to compute the corresponding statistics of failure for bending, torsion, and other stress states that involve only uniaxial tensile stresses. These applications have become well known and are widely used (Refs. C-2 through C-4).

In problems that involve biaxial or triaxial tensile stresses, the situation is more complicated. Weibull gave, without formal proof, a procedure for treating such problems and showed how to apply it in some simple cases. Some investigators have expressed doubts concerning the rigor of Weibull's treatment of polyaxial stress states (Refs. C-5 and C-6) and there are indications that these doubts were later shared by Weibull himself (Refs. C-7 and C-8). Since, in addition, rather tedious calculations are required for each polyaxial stress state, in practical structures that involve continuously varying stress states there is a natural tendency to use approximations. One simple approximation, which constitutes, in fact, the only technique suggested for handling polyaxial stress statistics in a recent treatise on fracture (Ref. C-9), is to assume that

$$P_s(\sigma_1, \sigma_2, \sigma_3) = P_s(\sigma_1)\, P_s(\sigma_2)\, P_s(\sigma_3) \tag{C-1}$$

where P_s is the probability of survival, and σ_1, σ_2, σ_3 are the principal stresses. This method was mentioned earlier by Barnett et al. (Ref. C-5). For brevity, it is referred to here as the Barnett-Freudenthal (BF) approximation.

One objective of the present investigation is to explore the limitations of this approach and thus assist potential users in understanding the general

nature and approximate magnitude of the errors involved. Another objective is to point out an alternative approximation that is preferred in some applications. For the sake of simplicity and brevity, interest is focused primarily on biaxial tension applied to isotropic materials whose fracture statistics in simple tension can be described with satisfactory accuracy by Weibull's two-parameter formulation:

$$P_f = 1 - \exp[-k\,\sigma^m] \qquad \text{(C-2)}$$

or

$$\ln P_s = \ln(1 - P_f) = -k\,\sigma^m \qquad \text{(C-3)}$$

where P_f is the probability of fracture. An exact treatment of the Weibull theory solution is also given.

The BF approximation would be strictly correct if all the cracks were oriented with their planes normal to any one of the principal stresses. However, this is rarely the case and certainly does not occur in isotropic materials. Normally, there will be some cracks inclined to the principal axes that will be fractured by combined stresses even though capable of surviving any of the principal stresses applied individually. As a result, Eq. (C-1) is usually unconservative.

In the case of equibiaxial tension, Eq. (C-1) reduces to

$$P_s(\sigma, \sigma) = [P_s(\sigma, o)]^2 \qquad \text{(C-4)}$$

or

$$\frac{\ln P_s(\sigma, \sigma)}{\ln P_s(\sigma, o)} = 2 \qquad \text{(C-5)}$$

Equation (C-5) implies that, for the small values of P_f of primary interest

$$\frac{\ln P_s(\sigma, \sigma)}{\ln P_s(\sigma, o)} = \frac{\ln [1 - P_f(\sigma, \sigma)]}{\ln [1 - P_f(\sigma, o)]} \cong \frac{P_f(\sigma, \sigma)}{P_f(\sigma, o)} = 2 \qquad \text{(C-6)}$$

i.e., the probability of failure in equibiaxial tension is twice that for uniaxial tension.

As noted earlier, it is clear, from general principles, that this is an underestimate of the probability of equibiaxial fracture. However, in order to determine the magnitude of the error involved, we must turn to theory. For this purpose, we select a recent reformation of weakest link theory in which the flaws are identified as cracks (Ref. C-10), and the polyaxial stress statistics are derived in a straightforward manner from the basic assumptions. In Ref. C-8, it was noted that when the probability of fracture in simple tension obeys Eq. (C-2), the probability for equibiaxial tension becomes

$$P_f(\sigma, \sigma) = 1 - \exp\left[- (2m + 1) k\sigma^m \frac{\Gamma(m)\,\Gamma(1.5)}{\Gamma(m + 1.5)}\right] \qquad \text{(C-7)}$$

For integral values of m, Eq. (C-7) becomes

$$P_f(\sigma, \sigma) = 1 - \exp\left[- \frac{k\,(m)\,!\,\sigma^m}{(m - 0.5)\,(m - 1.5)\,--\,0.5}\right] \qquad \text{(C-8)}$$

Combining Eqs. (C-2) and (C-8), we obtain

$$\frac{\ln P_s(\sigma, \sigma)}{\ln P_s(\sigma, o)} = \frac{m\,!}{(m - 0.5)\,(m - 1.5)\,---\,0.5} \qquad \text{(C-9)}$$

In Fig. C-1, this result is compared with the previous one given in Eq. (C-5), and we conclude that the BF approximation leads to the correct result for equibiaxial tension when m = 1. However, for m = 5 and 10, it underestimates small equibiaxial failure probabilities by factors of approximately 2 and 3, respectively. Note that Eq. (C-9) can be approximated surprisingly well by the very simple equation

$$\frac{\ln P_s(\sigma, \sigma)}{\ln P_s(\sigma, o)} \cong 2\,(m)^{0.45} \qquad \text{(C-10)}$$

In view of the expected unconservatism of the BF approximation; it is somewhat surprising that Fig. C-1 implies that it is correct for m = 1 and conservative for $m < 1$. The explanation may be that these are abnormal cases. With the use of Eq. (C-2), it is readily shown that the slope of the $P_f(\sigma)$ curve is zero at $\sigma = 0$ when $m > 1$, in agreement with observation. For m = 1, the slope is finite; for $m < 1$, it is infinite; such behavior is rarely, if ever, encountered in nature.

We now consider the way the size of the error in Eq. (C-1) varies with stress ratio, again focusing our attention on the biaxial case. It was pointed out earlier (Ref. C-8) that, for materials that obey Eq. (C-2), the Weibull theory and that of Batdorf and Crose (Ref. C-10) lead to the same results. In the general case of triaxial tension characterized by principal stresses σ_1, σ_2 and σ_3, where $\sigma_1 \geq \sigma_2 \geq \sigma_3 \geq o$, both theories imply that the probability of fracture takes the form

$$P_f = 1 - \exp\left[-k\int_o^{2\pi} d\phi \int_o^{\pi} \left(\sigma_1 \cos^2\phi \sin^2\theta + \sigma_2 \sin^2\phi \sin^2\theta + \sigma_3 \cos^2\theta\right)^m \sin\theta \, d\theta\right] \qquad \text{(C-11)}$$

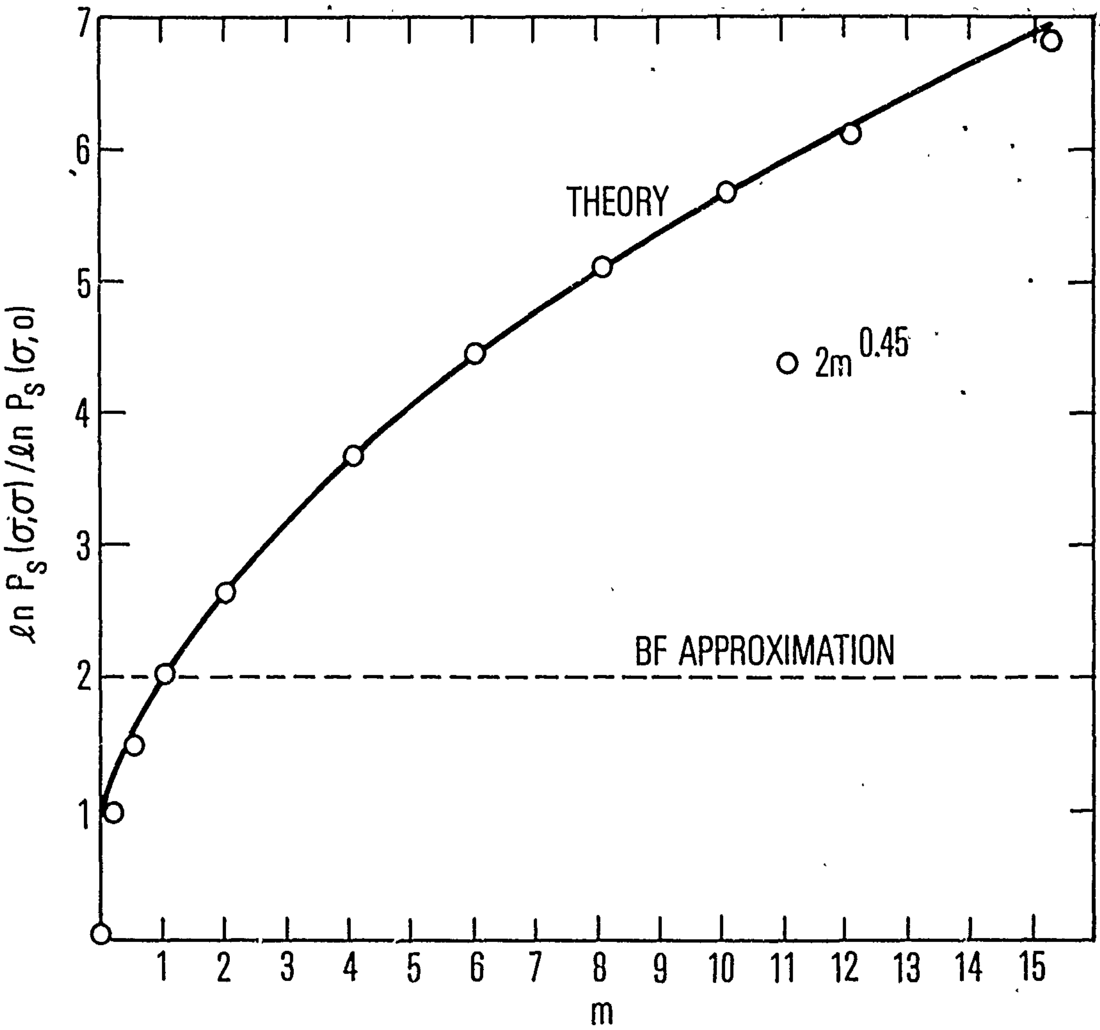


Figure C-1. Ratio of ℓn of Probability of Survival in Equibiaxial Tension to that in Uniaxial Tension [equal to $P_f(\sigma, \sigma)/P_f(\sigma, o)$ when $P_f \ll 1$]

where θ and ϕ are the polar and azimuthal angles, respectively. As noted by Shur (Ref. C-11) the integral in Eq. (C-11) leads to

$$\int_{o}^{2\pi} d\phi \int_{o}^{\pi} (\ )^{m}\, d\theta = 4\pi\, \sigma_{1}^{m} \frac{(m!)^{2}}{(2m+1)!} \sum_{i,j,k}^{m} \frac{(2i)!\,(2j)!\,(2k)!}{(i!\,j!\,k!)^{2}}\, s^{j}\, t^{k} \tag{C-12}$$

where i, j, k are any positive integers that satisfy the inequalities $o \leq i, j, k \leq m$ and $i + j + k = 1$, and where

$$s \equiv \sigma_2/\sigma_1 \tag{C-13a}$$

$$t \equiv \sigma_3/\sigma_1 \tag{C-13b}$$

For the biaxial case, $\sigma_3 = t = o$, and the probability of survival becomes

$$P_s = \exp\left[- \frac{4\pi k}{2m+1}\, \sigma_1^{m}\, F\,(m,\ s) \right] \tag{C-14}$$

where

$$F\,(m,\ s) \equiv \frac{(m!)^{2}}{(2m)!} \sum_{i=o}^{m} \frac{(2i)!\,[2(m-i)]!}{[i!\,(m-i)!]^{2}} \tag{C-15}$$

Thus, in the general biaxial case,

$$\frac{\ln P_s\,(\sigma_1,\ \sigma_2)}{\ln P_s\,(\sigma_1,\ o)} = \frac{F\,(m,\ s)}{F\,(m,\ o)} = F\,(m,\ s) \tag{C-16}$$

where m is any positive integer, and $o \leq s = \dfrac{\sigma_2}{\sigma_1} \leq 1$.

The BF approximation gives, for the biaxial case

$$\frac{\ell n\ P_s\ (\sigma_1,\ \sigma_2)}{\ell n\ P_s\ (\sigma_1,\ o)} = \frac{\ell n\ P_s\ (\sigma_1,\ s)}{\ell n\ P_s\ (\sigma_1,\ o)} = 1 + s^m \tag{C-17}$$

Curves that represent Eqs. (C-16) and (C-17) are shown in Fig. C-2. It is evident that, for values of m of practical interest, the BF approximation consistently underestimates the fracture contribution of the second stress and that the error is largest, in absolute terms, at least, for the equibiaxial case, $s = 1$.

Another way to compare theory with the BF approximation is to take the ratios of Eqs. (C-16) and (C-17). Thus,

$$\frac{\ell n\ (P_s)_{Th}}{\ell n\ (P_s)_{BF}} = \frac{F\ (m,\ s)}{1 + s^m} \tag{C-18}$$

This result is shown in Fig. C-3. It is evident that the ratio increases monotonically from $s = o$ to $s = 1$. Moreover, in the range $2 < m \lesssim 10$, the ratio is reasonably constant near $s = 1$.

The first of these observations suggests a simple method for obtaining an upper bound to the failure probability, or a lower bound to the survival probability of a biaxially stressed structure; viz., it implies that, in every element of the structure,

$$\frac{\ell n\ (P_s)_{Th}}{\ell n\ (P_s)_{BF}} = \frac{F\ (m,\ s)}{1 + s^m} \leq \frac{F\ (m,\ 1)}{2} \cong m^{0.45} \tag{C-19}$$

Since Eq. (C-19) holds for every element individually, it must hold for the structure as a whole.

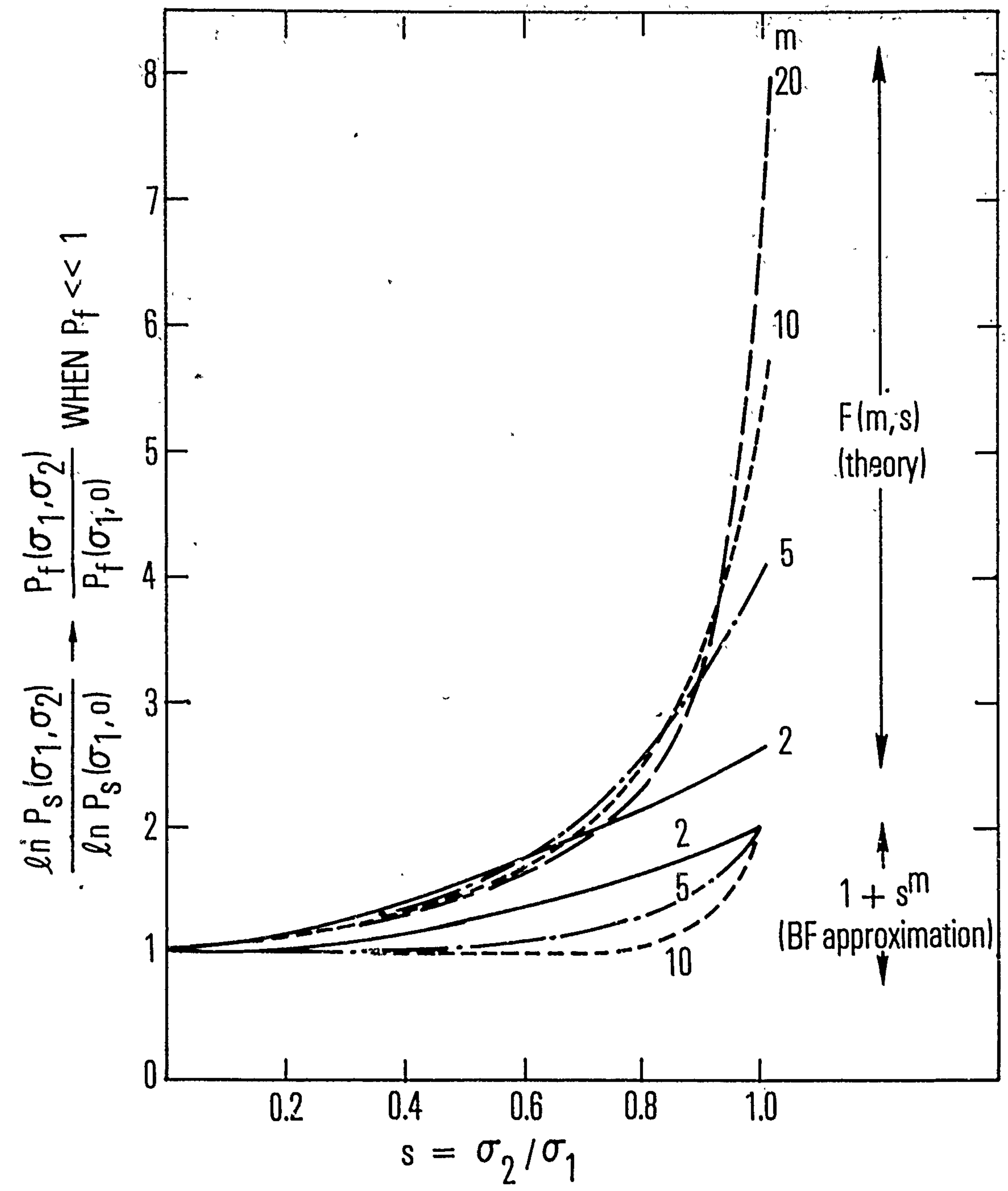


Figure C-2. Comparison of Theoretical Survival Prediction with Barnett-Freudenthal (BF) Approximation

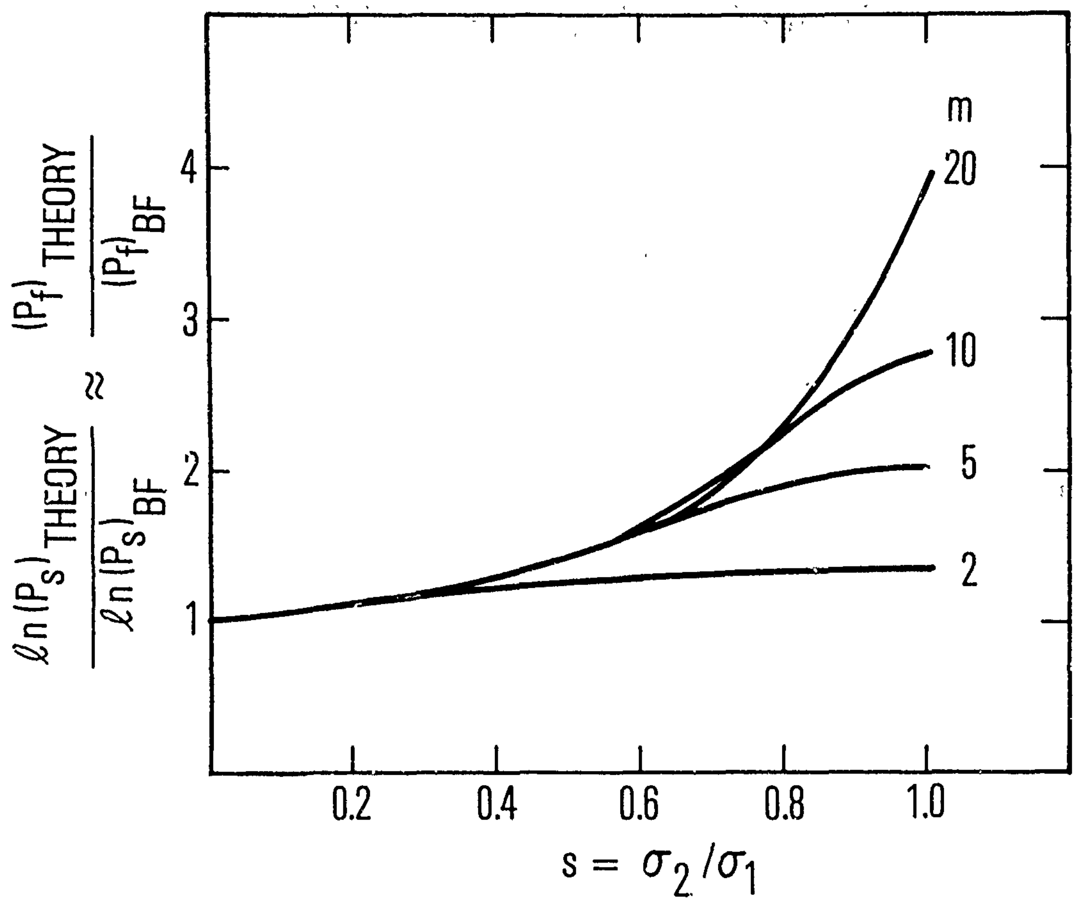


Figure C-3. Ratio of Theoretical Prediction to Barnett-Freudenthal (BF) Approximation

Thus, we conclude that

$$-\ell n\,(P_s)_{BF} \leq -\ell n\,(P_s)_{Th} \leq -m^{0.45}\,\ell n\,(P_s)_{BF} \qquad \text{(C-20)}$$

For small values of P_f, this implies that

$$(P_f)_{BF} \leq (P_f)_{Th} < m^{0.45}\,(P_f)_{BF} \qquad \text{(C-21)}$$

Because the curves of Fig. C-3 are fairly flat near $s = 1$, in the case of structures whose loading is nearly equibiaxial, the second inequality in Eqs. (C-20) and (C-21) approaches equality. In the case of circular disks with uniform lateral or concentric ring loading, for instance, the (small) probabilities of failure should, according to the theories discussed here, be quite close to $m^{0.45}$ times the result given by the BF approximation, which yields a simple closed-form solution.

As mentioned earlier, the preceding analysis is based on the assumption that the uniaxial tensile fracture statistics of the material under consideration can be described with adequate accuracy by Weibull's two-parameter equation. When the material is better represented by his three-parameter equation, the analysis becomes more complicated and is beyond the scope of the present paper. However, the general observation that Eq. (C-1) is unconservative still applies, and it is to be expected that the size of the error might be substantial, particularly when the Weibull parameter m is large. Finally, it should be pointed out that, for materials that obey Eq. (C-2), an accurate Weibull-type treatment of structural problems that involve biaxial or triaxial stresses is not difficult when a computing machine is used in conjunction with finite elements. This is because closed-form solutions to the integral in Eq. (C-11) are obtainable; these solutions are given in Eqs. (C-12), (C-14), and (C-15).

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APPENDIX D. GRAPHITE MODELING

A. GRAIN STRENGTH

We assume the body is composed of (1 - P)n identical grains of graphite and Pn pores; where n is the total number of pore and grain sites, and P is the fractional porosity. Each grain is assumed to have a critical stress S_c, the stress that will fracture the grain when applied in its weakest direction, i.e., parallel to the c axis. When simple tension is applied in some other direction, it is assumed that fracture occurs when the component of stress in the c direction reaches S_c. Since the stress in the c direction is then $\sigma \cos^2 \theta$, where σ is the applied stress, and θ is the angle between the stress axis and the c axis, it follows that each grain has a "strength" given by

$$\sigma_s = S_c / \cos^2 \theta \tag{D-1}$$

B. CRACK FORMATION

The model of Buch, Zimmer, and Meyer[1] assumes that there are chance aggregations of grains, all with c directions approximately parallel to the axis of tension, that occupy penny-shaped regions with planes normal to the tensile axis. Such arrays will be abnormally weak and will fracture prematurely; thus cracking will occur. As stress increases, the number and size of cracks increase also. Consider first the case of zero porosity.

We assume here that the capacity of an array to resist cracking is given by the average strength of its constituent grains. If the distribution of grain strengths in an array of N_o grains is the same as in the material as a whole, and the maximum angle between c axis and stress axis is θ_{max}, then the mean strength can be found as follows.

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[1] J. D. Buch, J. E. Zimmer, and R. A. Meyer, An Analytical Microstructural Model for the Fracture of Graphite, ATR-74(7425)-4, The Aerospace Corp., El Segundo, California (28 June 1974).

We designate the number of grains with an angle of inclination less than θ as $N(\theta)$. Since the orientation is random, the probability of a grain having an orientation between ϕ and $\phi + d\phi$ is the ratio of the corresponding solid angle, $2\pi \sin\phi \, d\phi$, to the solid angle of a hemisphere, i.e.,

$$P(\phi)\, d\phi = \sin\phi \, d\phi \tag{D-2}$$

Thus, the probability that the grain is inclined by less than θ is

$$\int_0^\theta P(\phi)\, d\phi = \int_0^\theta \sin\phi\, d\phi = 1 - \cos\theta \tag{D-3}$$

Since all N_o grains have an orientation angle less than θ_{max}, the number $N(\theta)$ is given by

$$N(\theta) = N_o \frac{1 - \cos\theta}{1 - \cos\theta_{max}} \tag{D-4}$$

Now, the average strength of the cracks is, by definition

$$\bar{\sigma} = \frac{1}{N_o} \int_0^{\theta_{max}} \sigma_s(\theta)\, dN \tag{D-5}$$

Inserting Eqs. (D-1) and (D-4) into (D-5) and carrying out the indicated operation, we obtain

$$\bar{\sigma} = S_c / \cos\theta_{max} \tag{D-6}$$

If P_c designates the probability that any arbitrarily chosen grain is inclined by less than θ_{max}, then, from Eqs. (D-3) and (D-6)

$$P_c = 1 - \cos \theta_{max} = 1 - S_c/\overline{\sigma} \qquad \text{(D-7)}$$

If we assume that all grains are randomly oriented and independent of the orientation of their neighbors, then the probability that <u>any</u> selected group of N grains have an orientation angle less than θ_{max} is given by

$$P_N = P_c^N = (1 - S_c/\overline{\sigma})^N \qquad \text{(D-8)}$$

In particular, Eq. (D-8) gives the probability that the N grains in a penny-shaped array centered at any designated site will have an average strength of $\overline{\sigma}$ or less. Since we have assumed that the strength of an array is equal to the average strength of its constituent grains, it follows that the probability that the array will fracture when the stress in the material is σ is given by

$$P_N = (1 - S_c/\sigma)^N \qquad \text{(D-9)}$$

We note in passing that in deriving Eq. (D-8), attention was focused exclusively on the grains comprising the array, and nothing was said about neighboring grains. Thus, the proper interpretation of the equation is that P_N is the probability that <u>N or more</u> grains have an average stress of $\overline{\sigma}$, i.e., that the crack formed contains <u>N or more grains.</u>

Up to this point, we have assumed that all sites were occupied by grains that are identical except for orientation. We now consider porosity.

Following the theory of Buch, Zimmer, and Meyer,[2] we assume that the array contains NP pores and N(1 - P) grains, where P is the volume fraction of porosity. The average strength of the array then becomes

$$\overline{\sigma'} = \frac{1}{N}\left[NP \cdot 0 + N(1 - P)\, S_c/\cos\theta_{max}\right]$$

$$= (1 - P)\, S_c/\cos\theta_{max} \qquad \text{(D-10)}$$

The probability that an array will be cracked at stress σ is equal to the probability that each of the N(1 - P) grains will be stressed at the limit of its strength, i.e.,

$$P'_N = \left[1 - (S_c/\sigma)_G\right]^{N(1-P)} \qquad \text{(D-11)}$$

where subscript G is inserted in order to emphasize that S_c and σ refer to the critical stress of an actual grain (not a site average) and the mean stress applied to an actual grain, respectively. If S'_c refers to critical stress averaged over all N sites, and σ to the average stress at all N sites (therefore to the average stress in the material), then

$$S'_c = (1 - P)\,(S_c)_G = \sigma_L \qquad \text{(D-12)}$$

where σ_L is the elastic limit, and

$$\sigma = \sigma_G\,(1 - P) \qquad \text{(D-13)}$$

[2] See Footnote 1.

Thus, P'_N can be written

$$P'_N = \left[1 - (\sigma_L/\sigma)\right]^{N(1-P)} \tag{D-14}$$

Now, in order to determine the number of cracks of size N or larger in a material that contains a total of nP pores and (1 - P)n grains, we multiply the probability of having a crack that contains N or more sites centered at a designated site P'_N by the total number of sites. In order to obtain the number of cracks that contain N sites but no more, we subtract the number of cracks that contain N + 1 or more sites. Thus, M_{Nn}, the number of cracks that contain N and only N sites in a unit volume of material with $\tilde{n}$ sites per unit volume is

$$M_{Nn} = \tilde{n}\,(P'_N - P'_{N+1}) = -\,\tilde{n}\,\frac{dP_N}{dN} \tag{D-15}$$

C. CRACK STRAIN

According to the model of Buch, Zimmer, and Meyer, the inelastic strain is caused by additional extension of the stressed material that results from the presence of cracks. In order to determine the stress-strain curve, it is necessary to know what strain will result from a given number of cracks per unit volume of size N uniformly distributed throughout an infinite medium. This strain is equal to the relative displacement of two points initially a unit distance apart on a line parallel to the stress axis. It can be shown that a crack of radius r located halfway between the points and with its plane normal to the stress axis will result in a relative displacement given by the expression

$$\frac{16}{3}\, r^3\, \frac{(1+\nu)(2-\nu)}{\pi/2}\, \frac{\sigma}{E} \tag{D-16}$$

Now

$$\pi r^2 = N g^2 \tag{D-17}$$

where g is the length of a grain edge (assumed square), so the strain will be proportional to $N^{3/2}$. Since a calculation of the strain averaged over all the positions at which one crack per unit volume might be located is not available, we simply assume an unknown proportionality constant K and obtain

$$\epsilon_c^N = K\,(\sigma/E)\,g^3\,N^{3/2} \tag{D-18}$$

K is an unknown constant that depends slightly on Poisson's ratio ν. It will be determined experimentally later.

D. DERIVATION OF STRESS-STRAIN RELATION

We can now compute the total crack strain, which is simply the sum of the contributions of all the cracks:

$$\epsilon_c = \sum_{N=1}^{N_f} M_{Nn}\,\epsilon_c^N \tag{D-19}$$

where N_f represents the number of sites in the largest crack at the time of fracture. Since, at fracture, there is only one crack of size N_f in the entire volume, and at lower stresses, there isn't even one, the ceiling on the upper limit can be removed with negligible error. Thus,

$$\epsilon_c = \sum_{N=1}^{\infty} M_{Nn}\,\epsilon_c^N \tag{D-20}$$

Substitution of Eqs. (D-14), (D-15), and (D-18) into Eq. (D-20) yields

$$\epsilon_c = \tilde{n}\, K\, (\sigma/E)\, g^3 \sum_{N=1}^{\infty} N^{3/2} \frac{d}{dN} \left(1 - \frac{\sigma_L}{\sigma}\right)^{N(1-P)} \tag{D-21}$$

This equation is simplified slightly by noting that

$$\tilde{n} = 1/g^3, \qquad \text{or} \qquad \tilde{n}\, g^3 = 1 \tag{D-22}$$

Now

$$\left(1 - \frac{\sigma_L}{\sigma}\right)^{N(1-P)} = \exp\left[- N(1 - P)\, \ell n \left(1 - \frac{\sigma_L}{\sigma}\right)^{-1}\right] \tag{D-23}$$

and

$$\frac{d}{dN}\left(1 - \frac{\sigma_L}{\sigma}\right)^{N(1-P)} = (1 - P)\, \ell n \left(1 \frac{\sigma_L}{\sigma}\right)^{-1} \exp\ - N(1 - P)\, \ell n \left(1 - \frac{\sigma_L}{\sigma}\right)^{-1} \tag{D-24}$$

By using Eqs. (D-22) and (D-24) and converting the sum to an integral, we obtain

$$\epsilon_c = K(\sigma/E)\, (1 - P) \left(1 - \frac{\sigma_L}{\sigma}\right)^{-1} \int_{0.5}^{\infty} N^{1.5} \exp\ - N(1 - P)\, \ell n \left(1 - \frac{\sigma_L}{\sigma}\right)^{-1} dN \tag{D-25}$$

The integral can be evaluated by making the substitution

$$\chi \equiv N (1 - P) \ln \left(1 - \frac{\sigma_L}{\sigma}\right)^{-1} \tag{D-26}$$

The result is

$$\epsilon_c = \frac{K (\sigma/E)}{\left[(1 - P) \ln \left(1 - \frac{\sigma_L}{\sigma}\right)^{-1} \right]^{1.5}} \int_{\chi_o}^{\infty} \chi^{1.5} \ell^{-\chi} \, d\chi \tag{D-27}$$

where

$$\chi_o = 0.5 \ (1 - P) \ln \left(1 - \frac{\sigma_L}{\sigma}\right)^{-1} \tag{D-28}$$

Now

$$\int_{\chi_o}^{\infty} \chi^{1.5} e^{-\chi} d\chi = \int_{o}^{\infty} \chi^{1.5} e^{-\chi} d\chi - \int_{o}^{\chi_o} \chi^{1.5} e^{-\chi} d\chi \tag{D-29}$$

The first term on the right-hand side is the gamma function $\Gamma(2.5) = 1.33$. In order to establish an upper limit to the second term approximately, we note that

$$\int_0^{\chi_o} \chi^{1.5} e^{-\chi} d\chi < \int_0^{\chi_o} \chi^{1.5} d\chi = \frac{\chi_o^{2.5}}{2.5}$$

$$= 0.2 \left[(1 - P) \ln \left(1 - \frac{\sigma_L}{\sigma} \right)^{-1} \right]^{2.5} \tag{D-30}$$

For a typical porosity P = 0.2, this upper limit is less than 11% of Γ (2.5) for any value of stress greater than 1.5 σ_L and less than 4% for any value of stress greater than 2 σ_L. For the sake of simplicity, therefore, we neglect the second term on the right-hand side of Eq. (D-29) and approximate Eq. (D-27) by

$$\epsilon_c = \frac{1.33\ K\sigma/E}{\left[(1 - P) \ln \left(1 - \frac{\sigma_L}{\sigma} \right)^{-1} \right]^{1.5}} \tag{D-31}$$

From Eq. (D-31), it follows that simple graphites with the same porosity and Poisson ratio will all obey the relation

$$\epsilon_c = \frac{\text{const. } (\sigma/E)}{\left[\ln \left(1 - \frac{\sigma_L}{\sigma} \right)^{-1} \right]^{1.5}} \tag{D-32}$$

with the same value of the constant in Eq. (D-32) applicable in each case.

In Fig. D-1, a stress-strain curve for POCO graphite is compared with the results of Eq. (D-32). At Southern Research Institute (SRI), E was determined to be 1.93×10^6 psi, and σ_L appears to be approximately 2.4 ksi. Thus, the constant can be determined by a single point on the stress-strain

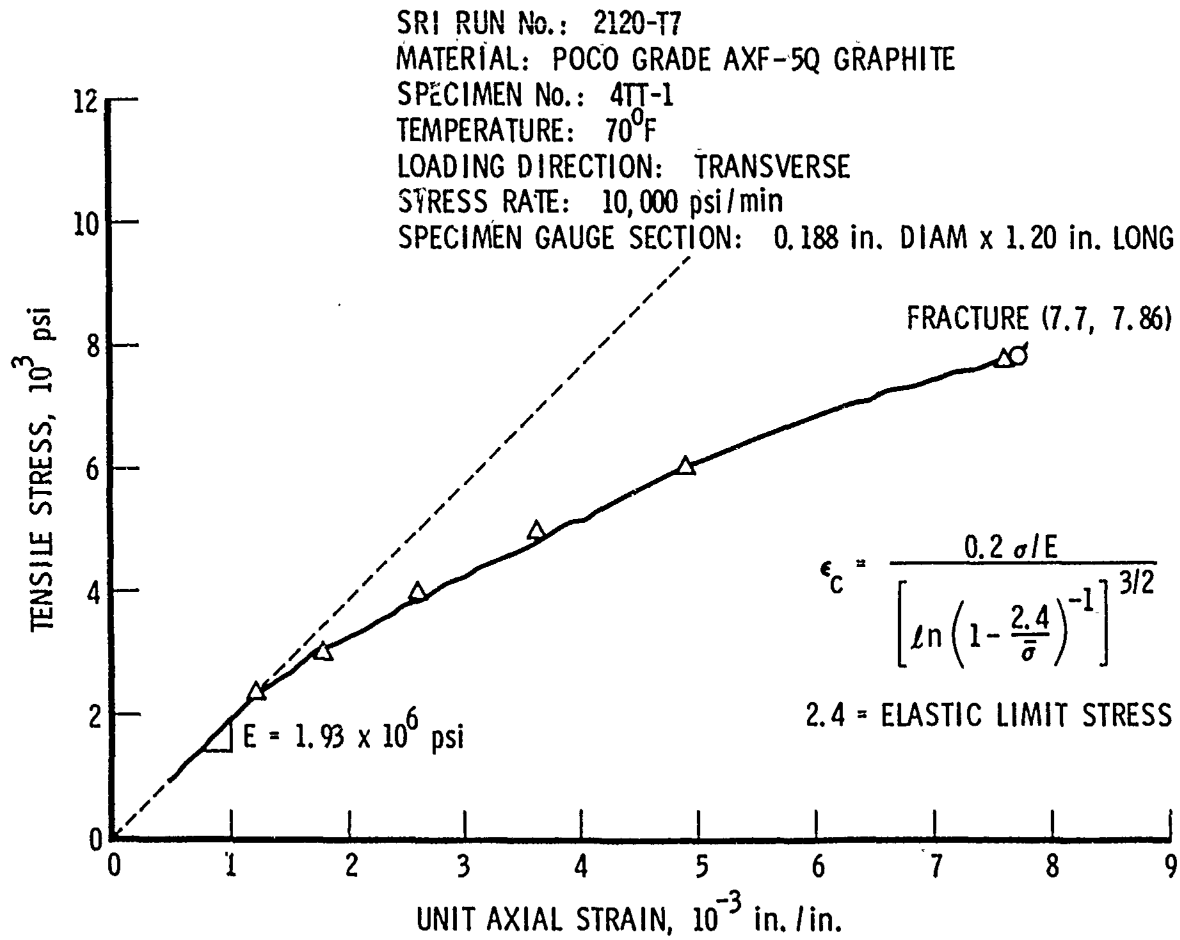

Figure D-1. Tensile Stress vs Axial Strain at 70°F for POCO Grade AX-5Q Graphite Specimen 4TT-1 from Billet 4(E2A-30A)

curve. The point $\sigma = 7.8$ ksi, $\epsilon_c = 3.6 \times 10^{-4}$ was chosen for this purpose, and a value of 0.2 was obtained for the unknown constant. The equation thus becomes

$$\epsilon_c = \frac{0.2\,(\sigma/E)}{\left[\ell n \left(1 - \frac{\sigma_L}{\sigma}\right)^{-1}\right]^{3/2}} \tag{D-33}$$

For convenience, the function

$$F\left(\frac{\sigma}{\sigma_L}\right) \equiv \frac{0.2}{\left[\ell n \left(1 - \frac{\sigma_L}{\sigma}\right)^{-1}\right]^{3/2}} \tag{D-34}$$

is plotted in Fig. D-2. This function is simply the ratio of the crack strain to the elastic strain.

In Eq. (D-33), it is implied that the crack strain is determined by applied stress and two material parameters, Young's Modulus and the elastic limit. The latter is sometimes difficult to determine accurately. Therefore, for purposes of checking the accuracy of the equation, it is sometimes preferable to determine what value of σ_L provides good agreement with the stress-strain curve near the fracture point, and then determine if this value falls in the range of values that would be estimated by different observers as the first appearance of nonlinearity in the curve. Solving Eq. (D-33) for σ_L, we obtain

$$\sigma_L = \sigma \left[1 - e^{-\left(\frac{\epsilon_c}{5\epsilon_e}\right)^{2/3}}\right] \tag{D-35}$$

where ϵ_e denotes elastic strain.

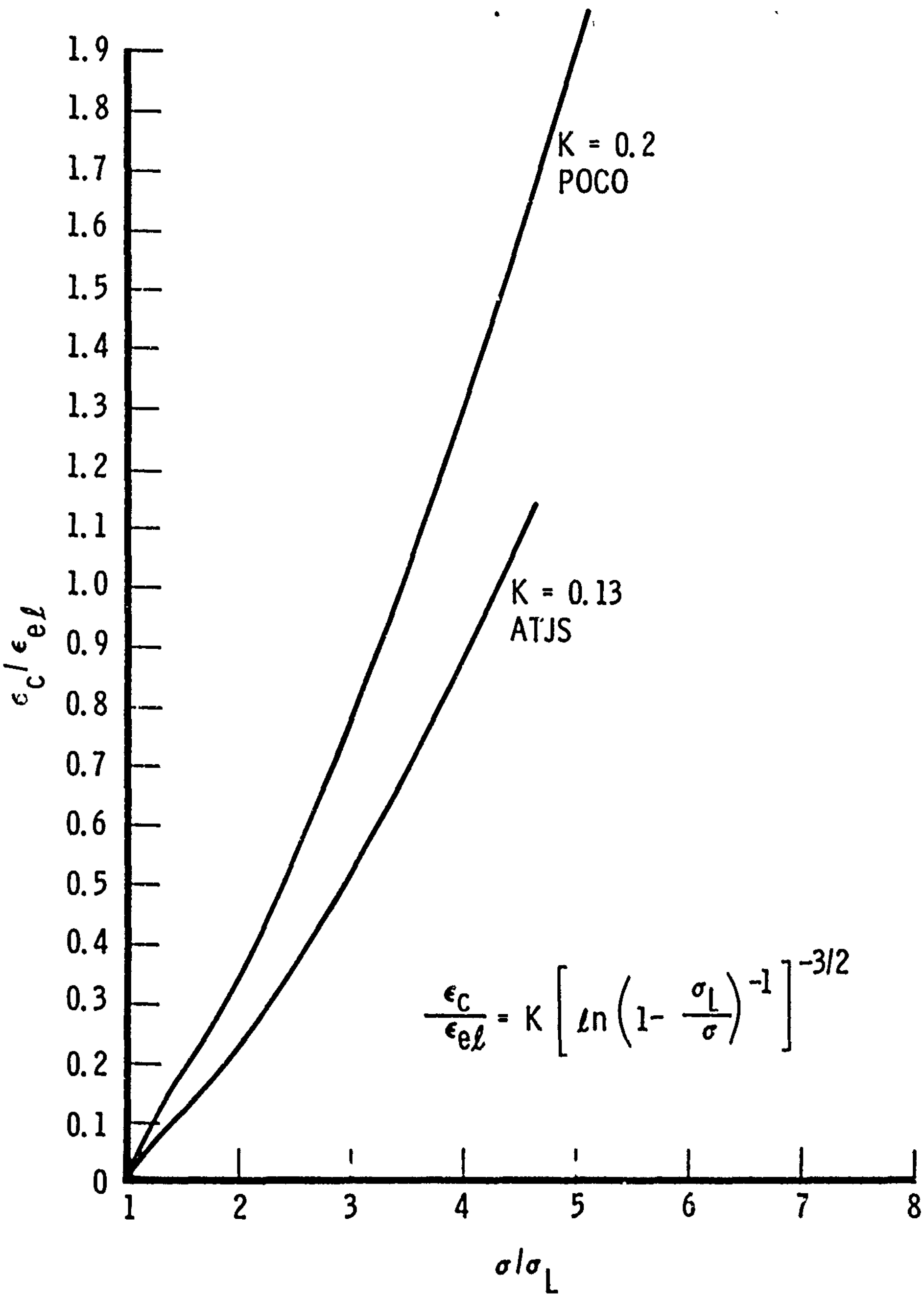

Figure D-2. Crack Strain vs Elastic Strain for POCO and ATJS Graphite

E. COMPARISON OF STRESS-STRAIN PREDICTION WITH EXPERIMENT

The constant 0.2 in Eq. (D-33) was chosen to give good agreement between theory and experiment at the fracture stress for a particular POCO graphite. The calculated points designated in Fig. D-1, however, indicate that the entire stress-strain curve is fitted very well by the equation. Figure D-3 is presented in order to check the accuracy of the equation against another POCO graphite with different values of E and σ_L. σ_L was determined by using Eq. (D-35). It is apparent that the agreement between theory and experiment is again very good.

POCO is a relatively simple graphite in that it is essentially isotropic and has approximately uniform grain size and pores approximately the same size as the grains. In Figs. D-4 and D-5, Eq. (D-33) is compared with test data for ATJS graphite, which is anisotropic and has variable pore and grain size. These complications are not taken into account in the theory underlying Eq. (D-33) and are beyond the scope of this report. The general shape of the curve appears to be predicted well, but, in some cases, the elastic limit is noticeably different from the value of σ_L that gives the observed crack strain at fracture. It is of some interest, therefore, that if the constant in Eq. (D-33) is changed from 0.2 to 0.13, very good agreement is obtained with experimental data on ATJS. It is possible that the differences between simple and complex graphites mainly affect the constant rather than the basic form of the equation that gives crack strain as a function of stress.

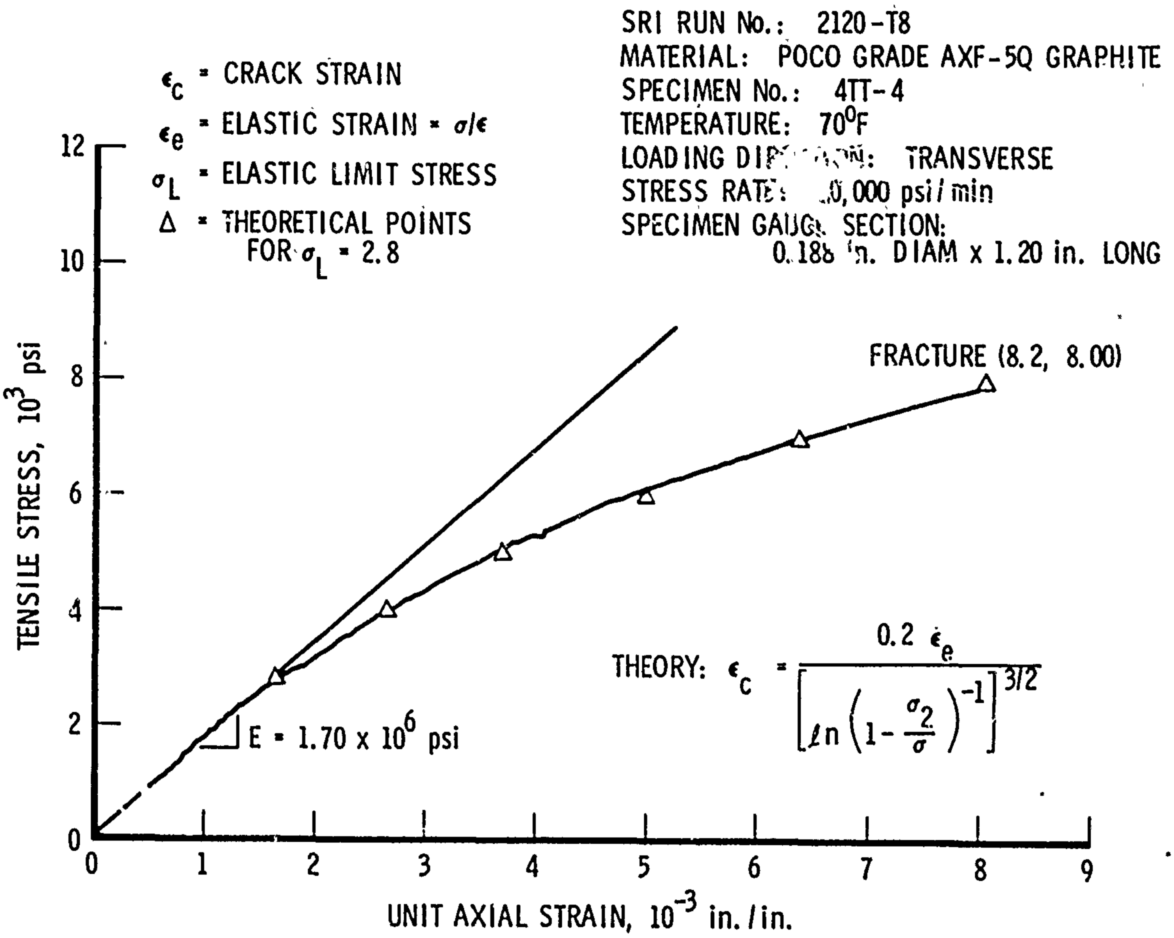

Figure D-3. Tensile Stress vs Axial Strain at 70°F for POCO Grade AXF-5Q Graphite Specimen 4TT-4 from Billet 4 (E2A-30A)

SRI RUN No.: 2139-T14-L16-7
MATERIAL: ATJS GRAPHITE
SPECIMEN No.: S-2-4
TEMPERATURE: 70°F
LOADING DIRECTION: WITH GRAIN
STRESS/STRAIN RATE: 10,000 psi/min
SPECIMEN GAUGE SECTION: 0.250 in. DIAM x 1.2 in. LONG

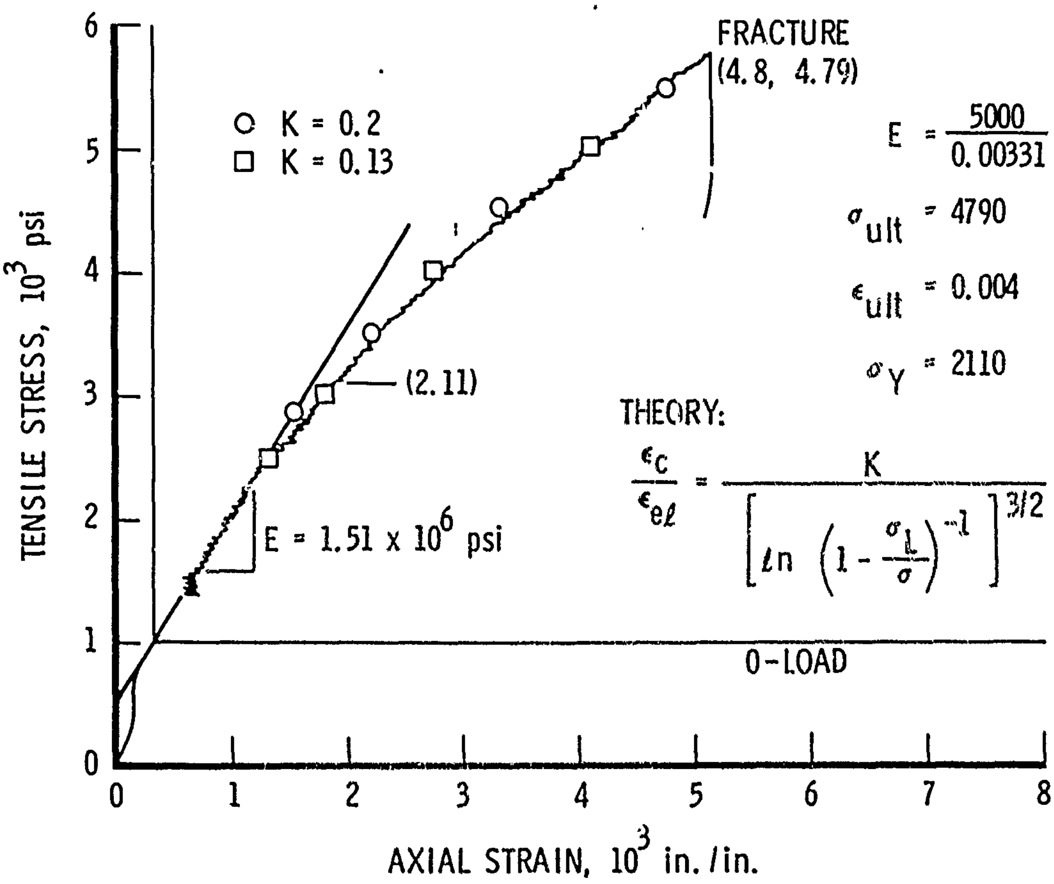

Figure D-4. Tensile Stress vs Axial Strain for ATJS Graphite Specimen S-2-4

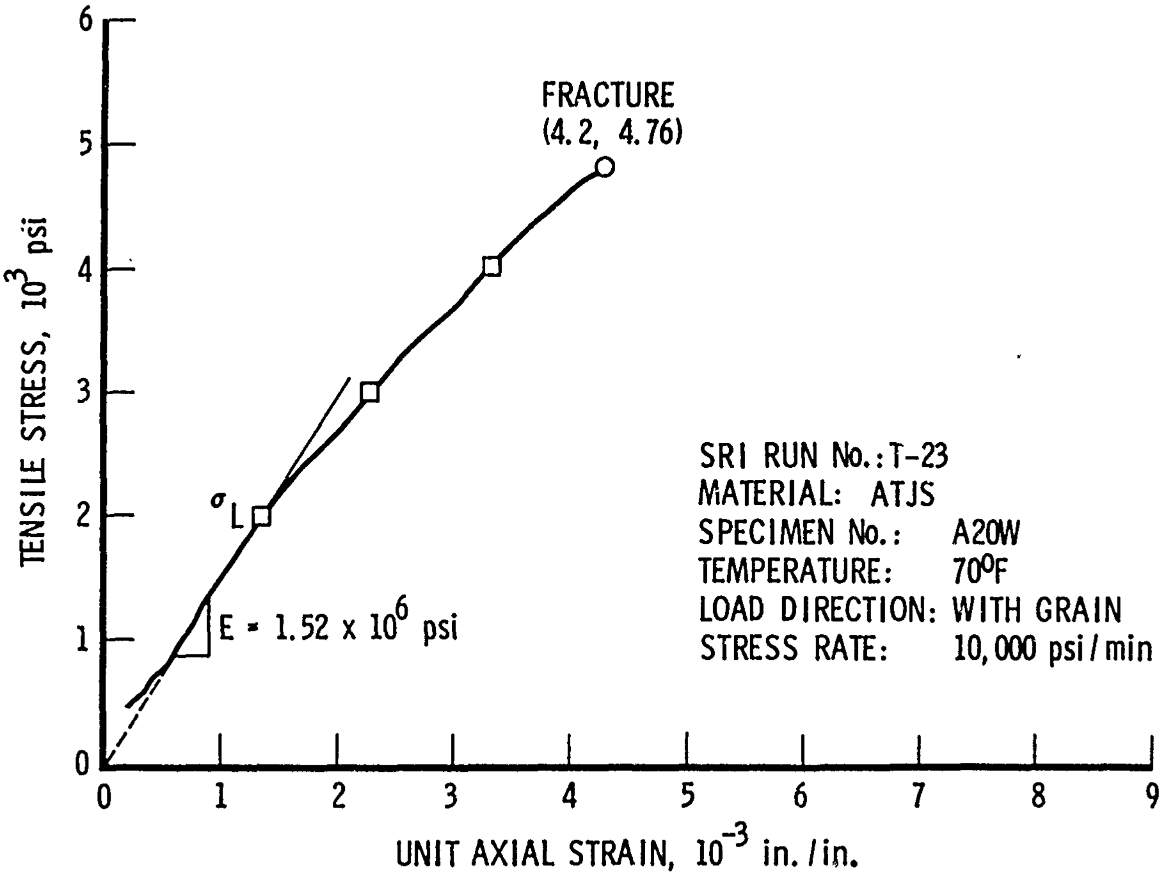

Figure D-5. Tensile Stress-Strain Curve for Specimen A20W from ATJS Billet L-1-8

APPENDIX E. THE THEORY OF RUNS APPLIED TO CRACK SIZE

(R. H. Huddlestone and D. C. Pridmore-Brown)

The theory of the distribution of cracked grains in a one-dimensional model that consists of a single row of grains, each with a probability p of being cracked, is identical with the theory of runs treated extensively in the statistical literature. First, a brief outline is given of those portions of the theory that are relevant to our problem, following the approach and notation of Fisz.[1] Then, these results are used to derive a formula for the probability of occurrence of a crack of a given size or greater. This formula, which we believe to be new, involves a final summation over a number of terms equal to the number of cracked grains divided by the crack length. Although it has not been possible to perform this summation analytically, an asymptotic expression is derived for it, which is then compared with a commonly used heuristic formula. Finally, some numerical comparisons between the exact and heuristic results are presented.

Consider a one-dimensional row of n grains, of which $n_1 = pn$ are cracked and $n_2 = n - n_1$ are uncracked. Then, there are $\binom{n}{n_1}$ ways in which the cracked grains can be distributed among the uncracked ones. Here, a run of i contiguous cracked grains is referred to as a crack of length i, and the total number of such cracks of length i is denoted by k_{1i}. Similarly, the number of runs of uncracked grains of length i is denoted by k_{2i}. Finally, the total number of cracks of all lengths is denoted by k_1, and of noncracks by k_2. Then

$$\Sigma k_{1i} = k_1 \quad ; \quad \Sigma k_{2i} = k_2$$
$$\Sigma i k_{1i} = n_1 \quad ; \quad \Sigma i k_{2i} = n_2 \tag{E-1}$$

[1] M. Fisz, Probability Theory and Mathematical Statistics, John Wiley, New York (1963).

where the summations are from 1 to n on the left and 1 to n_2 on the right.

Of the total number of configurations or ways of distributing n_1 cracked grains among $n_2 = n - n_1$ uncracked ones, vix., $\binom{n}{n_1}$, that number that contains exactly k_{11} cracks of length 1, k_{12} of length 2, . . . k_{1n} of length n_1, k_{2j} noncracks of length 1, . . . , k_{2n_2} noncracks of length n_2 . . . is just

$$N\left(k_{1i}, k_{2i}, n_1, n_2\right) = \begin{bmatrix} k_1 \\ k_{1i} \end{bmatrix} \begin{bmatrix} k_2 \\ k_{2i} \end{bmatrix} G\left(|k_1 - k_2|\right) \tag{E-2}$$

where $\begin{bmatrix} k_1 \\ k_{1i} \end{bmatrix}$ denotes the multinomial coefficient

$$\frac{k_1\,!}{k_{11}\,!\,k_{12}\,!\,.\,.\,.\,k_{1n_1}\,!}$$

which represents the number of ways that k_1 objects can be distributed among n_1 piles such that k_{1i} objects are in the ith pile, and $G(x) = 2$ for $x = 0$, 1 for $x = 1$, and 0 otherwise. $G\left(|k_1 - k_2|\right)$ accounts for the fact that the cracked and uncracked runs must alternate such that k_1 and k_2 can differ at most by one.

On the assumption that each of the $\binom{n}{n_1}$ arrangements of cracked and uncracked grains is equally probable, the ratio of Eq. (E-2) to $\binom{n}{n_1}$ becomes the conditional probability that there are k_{1i} cracks of length i, $i = 1, n_1$, and k_{2j} noncracks of length j, $j = 1, n_2$, given that there are n_1 cracked grains and n_2 uncracked grains

$$P\left(k_{1i}, k_{2i} \mid n_1, n_2\right) = N\left(k_{1i}, k_{2i}, n_1, n_2\right) / \binom{n}{n_1} \tag{E-3}$$

Now, we compute the probability that there is at least one crack of a given size, say J, or greater. Thus, we take Eq. (E-3), which contains n parameters (n_1 k_1's and $n_2 k_2'$s), and sum over all the k_2's and then sum separately over all the k_1's for $i < J$ and all the k_{1i} for $i \geq J$ such that we end up with an expression for the probability that contains the single parameter J. The first of these steps was carried out by Fisz with the identities

$$\sum \begin{bmatrix} k_2 \\ k_{2i} \end{bmatrix} = \binom{n_2 - 1}{k_2 - 1}$$

and

$$\sum \binom{n_2 - 1}{k_2 - 1} G\left(|k_1 - k_2|\right) = \binom{n_2 + 1}{k_1}$$

which are straightforward to derive. Thus,

$$N\left(k_{1i}, n_1, n_2\right) = \begin{bmatrix} k_1 \\ k_{1i} \end{bmatrix} \binom{n_2 + 1}{k_1} \qquad \text{(E-4)}$$

At this point, Fisz summed over the k_{1i} to obtain

$$N\left(k_1, n_1, n_2\right) = \binom{n_1 - 1}{k_1 - 1} \binom{n_2 + 1}{k_1}$$

which is the number of configurations with exactly k_1 cracks of any length. Instead, we sum over the long and the short cracks separately. Put

$$k_{1i} = S_{1i}, \qquad i < J$$
$$k_{1i} = \ell_{1i}, \qquad i \geq J$$

with

$$\Sigma\, s_{1i} = s_1,\ \Sigma\, \ell_{1i} = \ell_1,\ s_1 + \ell_1 = k_i$$

Then, it is clear that

$$\begin{bmatrix} k_1 \\ k_{1i} \end{bmatrix} = \begin{bmatrix} s_1 \\ s_{1i} \end{bmatrix} \begin{pmatrix} k_1 \\ s_1 \end{pmatrix} \begin{bmatrix} \ell_1 \\ \ell_{1i} \end{bmatrix}$$

Thus,

$$N\left(s_{1i}, \ell_{1i}, n_1, n_2\right) = \begin{bmatrix} s_1 \\ s_{1i} \end{bmatrix} \begin{pmatrix} k_1 \\ s_1 \end{pmatrix} \begin{bmatrix} \ell_1 \\ \ell_{1i} \end{bmatrix} \begin{pmatrix} n_2 + 1 \\ k_1 \end{pmatrix} \tag{E-5}$$

We now sum Eq. (E-5) over s_{1i} and ℓ_{1i}. Note first that

$$\left(x + x^2 + \ldots + x^{J-1}\right)^{s_1}$$

can be expanded on the one hand in a multinomial expansion

$$\sum \begin{bmatrix} s_1 \\ s_{1i} \end{bmatrix} x^{\Sigma i s_{1i}}$$

where the summation over all sets is s_{1i} such that $\Sigma s_{1i} = s_1$, and on the other hand in a binomial expansion

$$x^{s_1}\left(\frac{1 - x^{J-1}}{1 - x}\right)^{s_1} = x^{s_1}\left(1 - x\right)^{-s_1}\sum_{i=0}^{s_1}\binom{s_1}{i}(-1)^i x^{(J-1)i}$$

Similarly

$$\left(x^J + x^{J+1} + \ldots\right)^{\ell_1} = \sum\begin{bmatrix}\ell_1\\ \ell_{1i}\end{bmatrix} x^{\Sigma i\ell_{1i}} = x^{J\ell_1}\left(1 - x\right)^{-\ell_1}$$

thus

$$\left(x + x^2 + \ldots + x^{J-1}\right)^{s_1}\left(x^J + x^{J+1} + \ldots\right)^{\ell_1}$$

$$= \sum\begin{bmatrix}s_1\\ s_{1i}\end{bmatrix} x^{\Sigma is_{1i}} \cdot \sum\begin{bmatrix}\ell_1\\ \ell_{1i}\end{bmatrix} x^{\Sigma i\ell_{1i}}$$

$$= x^{s_1+J\ell_1}\left(1 - x\right)^{-k_1} \cdot \sum\binom{s_1}{i}(-1)^i x^{(J-1)i}$$

$$= x^{s_1+J\ell_1}\sum_{m=0}^{\infty}\binom{k_1 + m - 1}{m} x^m \cdot \sum_{i=0}^{s_1}\binom{s_1}{i}(-1)^i x^{(J-1)i} \qquad \text{(E-6)}$$

We now require

$$\Sigma i s_{1i} + \Sigma i \ell_{1i} = \Sigma i k_{1i}$$

to equal n_1 in the preceding expression. Then

$$\sum \begin{bmatrix} s_1 \\ s_{1i} \end{bmatrix} \begin{bmatrix} \ell_1 \\ \ell_{1i} \end{bmatrix}$$

when the summation is over all partitions $\left\{ s_{1i} \middle| i = 1, \ . \ . \ . \ J - 1 \right\}$ of $\binom{s_1}{n_1}$, and $\left\{ \ell_{1i} \middle| i = J, \ . \ . \ . \ n_1 \right\}$ of ℓ_1 is equal to the coefficient of x in the preceding expression. This coefficient is composed of those terms in the summation for which

$$s_1 + J\ell_1 + m + (J - 1)\, i = n_1$$

or

$$m = n_1 - s_1 - J\ell_1 - (J - 1)\, i$$

Thus

$$N\left(s_1, \ell_1, n_1, n_2 \right) = \binom{s_1 + \ell_1}{s_1} \binom{n_2 + 1}{s_1 + \ell_1} \sum_{i=0}^{s_1} \binom{s_1}{i} (-1)^i$$

$$\binom{n_1 - 1\, (J - 1)\, (\ell_1 + 1}{s_1 + \ell_1 - 1} \qquad \text{(E-7)}$$

or, equivalently

$$\left(N \quad k_1, s_1, n_1, n_2\right) = \binom{n_2+1}{k_1}\binom{k_1}{s_1}\sum_{i=0}^{s_1}\binom{s_1}{i}(-1)^i$$

$$\binom{n_1 - 1 - (J-1)(\ell_1 + i)}{k_1 - 1}$$

It does not appear possible to perform the summation over i. However, a summation over k_1 can be carried out as follows. First, repeated use of the binomial identity

$$\binom{A}{B}\binom{B}{C} = \binom{A}{C}\binom{A-C}{B-C}$$

permits one to reduce the number of terms that contain k_1. Thus

$$\binom{n_2+1}{k_1}\binom{k_1}{s_1}\binom{s_1}{i} = \binom{n_2+1}{k_1}\binom{k_1}{i}\binom{k_1 - i}{k_1 - s_1}$$

$$= \binom{n_2+1}{i}\binom{n_2+1-i}{k_1-i}\binom{k_1-i}{\ell_1}$$

$$= \binom{n_2+1}{i}\binom{n_2+1-i}{\ell_1}\binom{n_2+1-i-\ell_1}{k_1-\ell_1-i}$$

Thus

$$N\left(k_1, \ell_1, n_1, n_2\right) = \sum (-1)^i \binom{n_2+1}{i} \binom{n_2+1-i}{\ell_1} \binom{n_2+1-i-\ell_1}{k_1-\ell_1-i}$$

$$\binom{n_1 - 1 - (J-1)(\ell_1 + i)}{k_1 - 1} \qquad \text{(E-8)}$$

We now sum over k_1 by using the identity

$$\sum_j \binom{A}{B+j} \binom{C}{j} = \binom{A+C}{A-B}$$

Note that k_1 can run from $k_1 = 1$ to $k_1 = n_2 + 1$, which is the upper limit obtained when each uncracked grain acts as a crack separator. Thus

$$N\left(\ell_1 n_1, n_2\right) = \sum_{i=0}^{n_2+1-\ell_1} (-1)^i \binom{n_2+1}{i} \binom{n_2+1-i}{\ell_1} \binom{n - J(\ell_1 + i)}{n_2} \qquad \text{(E-9}$$

If we now sum over all ℓ_1, starting from $\ell_1 = 0$ and going to the natural upper limit, we expect to find

$$N\left(n_1, n_2\right) = \binom{n_1 + n_2}{n_1}$$

Thus, it is concluded that

$$\sum_{i=0} (-1)^i \binom{n_2+1}{i} \sum_{\ell_1=0} \binom{n_2+1-i}{\ell_1} \binom{n - J(\ell_1 + i)}{n_2} = \binom{n}{n_2}$$

and hence

$$N\left(\ell_1 \geq 1,\ n_1,\ n_2\right) = N\left(n_1,\ n_2\right) - N\left(\ell_1 = 0,\ n_1,\ n_2\right)$$

$$= \binom{n}{n_2} - \sum_{i=0} (-1)^i \binom{n_2+1}{i} \binom{n - Ji}{n_2}$$

$$= \sum_{k=1} (-1)^i \binom{n_2+1}{i} \binom{n - Ji}{n_2} \tag{E-10}$$

This expression, which is henceforth denoted by $N(J,\ n_1,\ n_2)$, represents the number of configurations that have at least one crack of length J or greater.

We now assert that

$$\sum_{i=0}^{n_2+1} (-1)^i \binom{n_2+1}{i} \binom{n - Ji}{n_2} = 0$$

This can be proved by using the identities

$$\binom{n_2+1}{i} = \binom{n_2}{i} + \binom{n_2}{i-1}$$

and

$$\sum_{i=0}^{n} (-1)^{n+i} \binom{n}{i} \binom{p+qi}{n} = q^n$$

the second of which has been proved by Riordon.[2] From this result, it follows that expression (E-10) is equal to $\binom{n}{n_2}$ when $J \leq [n/n_{2+1}]$ and that, otherwise, the upper limit on the summation may be taken to be $[n_1/J]$, where the square brackets denote integer part of.

Dividing through by $\binom{n}{n_2}$, we find, for the probability of at least one long crack (of size $\geq$ J)

$$P\left(J \middle| n_1, n_2\right) = \sum_{i=1}^{[n_1/J]} (-1)^{i+1} \binom{n_2+1}{i} \binom{n - J_i}{n_2} \Big/ \binom{n}{n_2}$$

$$= 1 \text{ for } J \leq [n_1/n_2 + 1] \qquad \text{(E-11)}$$

The corresponding unconditional probability is

$$P\left(J, n_1, n_2\right) = P\left(J \middle| n_1, n_2\right) \binom{n}{n_1} P^{n_1} (1-p)^{n_2} \qquad \text{(E-12)}$$

Strictly speaking, this expression should be summed over n_1. Instead, we believe it is sufficient to retain expression (E-11) and merely replace n_1 and n_2 with their expected values pn and qn, respectively, where $q = 1 - p$.

---

[2] J. Riordan, Combinatorial Identities, John Wiley, New York (1968), p. 51.

We now derive a closed-form analytical expression for Eq. (E-11) that is valid asymptotically as $n \to \infty$, i.e., for large samples.

First, expression (E-11) is rewritten as

$$P(J) = \sum \frac{(-1)^{i+1}}{i!} (nq+1)^{(i)} \prod_{k=0}^{qn-1} \left(1 - \frac{Ji}{n-k}\right)$$

which follows directly from the definition of the binomial coefficients. Here

$$(nq+1)^{(i)} = \prod_{k=1}^{2-i} (nq+k)$$

$$= \exp \Sigma \, \ell n \, (nq+k)$$

$$\simeq \exp \int_{1-i}^{1} \ell n \, (nq+k) \, dk$$

$$= (nq)^{i} \, [1 + O\,(1/n)]$$

Simularly,

$$\prod_{k=0}^{nq-1} \left(1 - \frac{Ji}{n-k}\right) = \exp \Sigma \, \ell n \left(1 - \frac{Ji}{n-k}\right)$$

$$\simeq \exp \int_0^{nq} \ell n \left(1 - \frac{Ji}{n-k}\right) dk$$

$$= \left(1 - \frac{Ji}{n}\right)^n \left(1 - \frac{Ji}{pn}\right)^{-pn} \left(\frac{pn - Ji}{n - Ji}\right)^{Ji}$$

$$= e^{-Ji} \cdot e^{Ji} \cdot p^{Ji} \left(\frac{1 - Ji/np}{1 - Ji/n}\right)^{Ji} \left[1 + O\left(\frac{1}{n}\right)\right]$$

Thus, as $n \to \infty$

$$P(J) \to \sum_{i=1}^{[pn/J]} \frac{(-1)^{i+1}}{i\,!} (nq\,p^J)^i \left(\frac{1 - Ji/np}{1 - Ji/n}\right)^{Ji} \qquad \text{(E-13)}$$

Expression (E-13) represents a finite sum of terms of alternating sign. For large n, the partial sums oscillate with increasing amplitude until a value of i is reached that is comparable to $e^{nq\,p^J}$. After this, the ratio of successive terms is less than unity, and the partial sums converge rapidly. Thus, if the upper limit [pn/J] is much greater than this critical value of i, i.e., if

$$pn/J \gg e^{nq\,p^J}$$

or (E-14)

$$q^J\,p^{J-1} \ll 1/e$$

we need not sum to the upper limit, but can stop at a value i_{max}, where

$$nq\, p^J \ll i_{max} \ll np/J$$

But over this range of i, viz., i = 1, i_{max}, the quantity

$$\left[\frac{1 - Ji/np}{1 - Ji/n}\right]^{Ji}$$

can be replaced by 1. After this replacement is made, we can let $i_{max} \to \infty$ and obtain

$$P\,(J) \to 1 - \exp\,(-\,nq\, p^J)$$

$$\simeq 1 - \exp\left[nq\, \ell n\, (1 - p^J)\right]$$

$$\doteq 1 - (1 - p^J)^{nq} \qquad \text{(E-15)}$$

which is valid as $n \to \infty$ under the weak restriction [Eq. (E-14)].

LABORATORY OPERATIONS

The Laboratory Operations of The Aerospace Corporation is conducting experimental and theoretical investigations necessary for the evaluation and application of scientific advances to new military concepts and systems. Versatility and flexibility have been developed to a high degree by the laboratory personnel in dealing with the many problems encountered in the nation's rapidly developing space and missile systems. Expertise in the latest scientific developments is vital to the accomplishment of tasks related to these problems. The laboratories that contribute to this research are:

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Chemistry and Physics Laboratory: Atmospheric reactions and atmospheric optics, chemical reactions in polluted atmospheres, chemical reactions of excited species in rocket plumes, chemical thermodynamics, plasma and laser-induced reactions, laser chemistry, propulsion chemistry, space vacuum and radiation effects on materials, lubrication and surface phenomena, photosensitive materials and sensors, high precision laser ranging, and the application of physics and chemistry to problems of law enforcement and biomedicine.

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Materials Sciences Laboratory: Development of new materials; metal matrix composites and new forms of carbon; test and evaluation of graphite and ceramics in reentry; spacecraft materials and electronic components in nuclear weapons environment; application of fracture mechanics to stress corrosion and fatigue-induced fractures in structural metals.

Space Physics Laboratory: Atmospheric and ionospheric physics, radiation from the atmosphere, density and composition of the atmosphere, aurorae and airglow; magnetospheric physics, cosmic rays, generation and propagation of plasma waves in the magnetosphere; solar physics, studies of solar magnetic fields; space astronomy, x-ray astronomy; the effects of nuclear explosions, magnetic storms, and solar activity on the earth's atmosphere, ionosphere, and magnetosphere; the effects of optical, electromagnetic, and particulate radiations in space on space systems.

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